# 國立成功大學105學年度碩士班招生考試試題

系　所：數學系應用數學碩士班

考試科目：高等微積分　　　　考試日期：0228，節次：2



※ 考生請注意：本試題不可使用計算機。　請於答案卷(卡)作答，於本試題紙上作答者，不予計分。

(1) (18 Points) True or false. If the statement is true, give a short proof; if the statement is false, give a counterexample.

(a) (6 Points) Any intersection of open subsets of $\mathbb{R}^n$ is open in $\mathbb{R}^n$.

(b) (6 Points) Let $A$ and $B$ be two subsets of $\mathbb{R}^n$. Then $\mathrm{cl}(A \cup B) = \mathrm{cl}(A) \cup \mathrm{cl}(B)$ where $\mathrm{cl}(A)$ is the closure of $A$ in $\mathbb{R}^n$.

(c) (6 Points) Let $f : [0,1] \to \mathbb{R}$ be a continuous function. Define a sequence $(a_n)$ of real numbers by $a_n = f\left(\frac{1}{n}\right)$ for $n \geq 1$. Then $\lim_{n\to\infty} a_n = f(0)$.

(2) (10 Points) Let $f : (0, \infty) \to \mathbb{R}$ be the function

$$f(x) = \frac{1}{x^2}, \quad x > 0.$$

Prove or disprove that $f$ is uniformly continuous on $(0, \infty)$.

(3) (15 Points)

(a) (7 Points) Prove that series of functions

$$f(x) = \sum_{n=1}^{\infty} \frac{\sin nx}{2^n}$$

converges uniformly on $[0, 2\pi]$.

(b) (8 Points) Evaluate $\int_0^{2\pi} f(x)^2 dx$.

(4) (12 Points) The Euclidean norm of a vector $x = (x_1, \cdots, x_n)$ in $\mathbb{R}^n$ is

$$\|x\| = \sqrt{x_1^2 + \cdots + x_n^2}.$$

If $S$ is a nonempty subset of $\mathbb{R}^n$ and $x_0$ is a point of $\mathbb{R}^n$ such that $x_0 \notin S$, the distance from $x_0$ to $S$ is defined to be

$$d(x_0, S) = \inf\{\|x_0 - y\| : y \in S\}.$$

Let $K$ be a (nonempty) closed and bounded subset of $\mathbb{R}^n$.

(a) (5 Points) Let $f : K \to \mathbb{R}$ be the function $f(x) = \|x_0 - x\|$ for $x \in K$. Prove that $f$ is continuous (use $\epsilon - \delta$ definition).

(b) (7 Points) Prove that there exists $y_0 \in K$ so that $d(x_0, K) = \|x_0 - y_0\|$. [Hint: Use (4a).]

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(5) (15 Points) Let $U$ be an open subset of $\mathbb{R}^n$. A function $f : U \to \mathbb{R}^m$ is differentiable at $x_0 \in U$ if there exists a linear map $T : \mathbb{R}^n \to \mathbb{R}^m$ such that

$$\lim_{h \to 0} \frac{\|f(x_0 + h) - f(x_0) - T(h)\|}{\|h\|} = 0.$$

In this case, the linear map $T$ is denoted by $(Df)(x_0)$ and called the total derivative of $f$ at $x_0$. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be the map

$$f(x, y) = (x - y^2, x^2 + y^2).$$

(a) (10 Points) Use definition to show that $f$ is differentiable at $(1, 1)$.
(b) (5 Points) Find the matrix representation of the linear map $(Df)(1, 1)$ with respect to the standard basis for $\mathbb{R}^2$.

(6) (15 Points)
(a) (7 Points) State the implicit function theorem.
(b) (8 Points) Can the equation

$$(x^2 + y^2 + z^2)^{1/2} = \cos(xz)$$

be solved uniquely for $y$ in terms of $(x, z)$ near $(0, 1, 0)$?

(7) (15 Points) Use Green's Theorem to evaluate the contour integral

$$\oint_C -x^2 y dx + xy^2 dy$$

where $C$ is the circle $x^2 + y^2 = 1$ going couter-clockwise.