國立成功大學106學年度碩士班招生考試試題

系 所：數學系應用數學

考試科目：線性代數

考試日期：0214，節次：1

※ 考生請注意：本試題不可使用計算機。 請於答案卷(卡)作答，於本試題紙上作答者，不予計分。

Notice. In the following problems, the symbols $\mathbf{R}$, $\mathbf{R}^{m\times 1}$, and $\mathbf{R}^{m\times n}$ are reserved for the fields of all real numbers, the set of real column vectors of size m, and the set of real $m\times n$ matrices, respectively.

Problems.

1. [10 points] Let $P_3 = \text{span}\{1, x, x^2, x^3\}$ be the set of all polynomials with real coefficients of degree less than or equal to three with the inner product
$$< f(x), g(x) >= \int_{-1}^{1} f(x)g(x)dx.$$
Find an orthonormal basis of P_3.

2. [10 points] Let A be a real $n\times n$ matrix with eigenvalues $\lambda_1 < \ldots < \lambda_n$ and its corresponding left and right eigenvectors u_i and v_i for $i = 1, \ldots, n$, respectively, i.e., $u_i^\top A = \lambda_i u_i^\top$ and $Av_i = \lambda_i v_i$ for $i = 1, \ldots, n$. Show that $\sum_{i=1}^{n} v_i u_i^\top$ is a nonsingular matrix.

3. [10 points] Let V be a vector space over $\mathbf{R}$ with an ordered basis $\alpha = \{1, x\}$ and let T be the linear operator on V defined by
$$T(1) = -5 + 4x \text{ and } T(x) = -9 + 7x.$$
Find the Jordan canonical form and a Jordan canonical basis for T.

4. [30 points] Suppose $V = \mathbf{R}^{2\times 2}$. Let V_1 and V_2 be two subspaces of V defined by
$$V_1 = \{\begin{bmatrix} a+b & 2a+3b \\ b & b \end{bmatrix} | a, b \in \mathbb{R}\} \text{ and } V_2 = \{\begin{bmatrix} 0 & a \\ -a+2b & b \end{bmatrix} | a, b \in \mathbb{R}\}.$$
   (a) [10 points] Determine the dimensions of V_1 and V_2.
   (b) [10 points] Determine the dimension of $V_1 + V_2$.
   (c) [10 points] Determine the dimension of $V_1 \cap V_2$.

5. [20 points]
   (a) [10 points] Let $T : \mathbf{R}^6 \to \mathbf{R}^3$ be a linear map. Prove or disprove that the dimension of the null space of T must be larger than or equal to 3.
   (b) [10 points] Let V_1 and V_2 be two subspaces of $\mathbf{R}^6$ such that $\dim(V_1) = 4$ and $\dim(V_2) = 3$. Prove or disprove that V_1 and V_2 have a nonzero vector in common.

6. [10 points] Let $\mathbf{v}_1 = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix}^\top$, $\mathbf{v}_2 = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix}^\top$ and $\mathbf{v}_3 = \begin{bmatrix} 0 & 1 & 1 \end{bmatrix}^\top$ be a basis of the vector space $\mathbf{R}^3$. Let $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\} \subseteq \mathbf{R}^3$ be a set of vectors satisfying
$$\mathbf{u}_i^\top \mathbf{v}_j = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j. \end{cases}$$
Let $\mathbf{w} = \begin{bmatrix} 2 & 1 & 3 \end{bmatrix}^\top$. Compute the products $< \mathbf{w}, \mathbf{u}_1 >$, $< \mathbf{w}, \mathbf{u}_2 >$, and $< \mathbf{w}, \mathbf{u}_3 >$.

7. [10 points] Let $P_2 = \text{span}\{1, x, x^2\}$ be the set of all polynomials with real coefficients of degree less than or equal to two and T be a linear operator on P_2 defined by $T(p) = p'$, the derivative of p. Find the minimal polynomial for T.