※ 考生請注意：本試題不可使用計算機。 請於答案卷(卡)作答，於本試題紙上作答者，不予計分。

1. (15%)
   (a) (5%) State the Intermediate Value Theorem.
   (b) (10%) Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function and I be an interval. Prove that $f(I)$ is an interval.

2. (10%) Show that if f is continuous, then

$$\int_0^x f(u)(x-u)\,du = \int_0^x \int_0^u f(t)\,dtdu.$$

(Hint: Use the Fundamental Theorem of Calculus)

3. (10%) Find all points in $\mathbb{R}$ where the series

$$\sum_{n=0}^{\infty} \frac{(x-1)^n}{2^n+1}$$

converges (that is, the interval of convergence).

4. (10%) Let $R \subset \mathbb{R}^2$ be the region bounded by the lines $x+y=1$, $x+y=-1$, $x-y=1$ and $x-y=-1$. Evaluate the integral

$$\iint_R \left(\frac{x-y}{x+y+2}\right)^2 dxdy$$

(Hint: Set $u = x+y$ and $v = x-y$, and use the change of variables.)

5. (10%) Find the 2nd-order Taylor polynomial of $f(x,y) = e^{x^2+y}$ about $(x,y) = (0,0)$.

6. (20%) Determine whether the following statements are true or false. If it is true, prove it. Otherwise, disprove it or give an counterexample.
   (a) (6%) Let X be a metric space and $f : \mathbb{R}^n \to X$ be a continuous map. Suppose that E is a closed and bounded subset of $\mathbb{R}^n$. Then $f(E)$ is a closed and bounded subset in X.
   (b) (7%) Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of integrable functions on $[a,b]$ and converges uniformly to a function f on $[a,b]$. Then

$$\lim_{n\to\infty} \int_a^b f_n(x)\,dx = \int_a^b f(x)\,dx$$

   (c) (7%) Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of nonnegative numbers. Suppose that $\sum_{n=1}^{\infty} \frac{a_n}{1+a_n}$ converges. Then $\sum_{n=1}^{\infty} a_n$ converges.

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7. (10%) Let $E = [a,b] \times [c,d]$ be a subset in $\mathbb{R}^2$. and $f : E \to E$ be a function satisfying

$$\|f(x) - f(y)\| < \|x - y\| \qquad \forall x, y \in E,\ x \neq y$$

where $\|\cdot\|$ is the usual metric in $\mathbb{R}^2$. Prove that there is a point $x_0 \in E$ such that $f(x_0) = x_0$.

8. (15%) Define $f : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$f(x,y) = \left(e^{2x+y},\ 4x^2 + 4xy + y^2 + 6x + 4y\right).$$

Define $U := f(\mathbb{R}^2)$ be the range of f.

(a) (7%) Prove that the inverse function of f (say $f^{-1} : U \to \mathbb{R}^2$) exists.

(b) (8%) Find the matrix representation of $\big(Df\big)(0,0)$.