國立成功大學108學年度碩士班招生考試試題

系 所：數學系應用數學

考試科目：高等微積分

考試日期：0224，節次：2

※ 考生請注意：本試題不可使用計算機。 請於答案卷(卡)作答，於本試題紙上作答者，不予計分。

1. (10 points) Determine the range of x on which the function

$$f(x) = \int_0^{x^2-x} e^{t^2-1}\, dt$$

is decreasing.

2. (10 points) Given a set X, two metrics d_1, d_2 on X are called *equivalent* if there exist $C_1, C_2 > 0$ such that

$$C_1 d_1(x,y) \le d_2(x,y) \le C_2 d_1(x,y)$$

for all $x, y \in X$. Prove that a subset $E \subset X$ is open with respect to d_1, if and only if it is open with respect to d_2.

3. Given a sequence $\{a_n\}$, define $\limsup_n a_n$ to be the supremum of all the limits of convergent subsequences of $\{a_n\}$. That is,

$$\limsup_n a_n = \sup\{c \mid \exists \{a_{n_i}\}_i \subset \{a_n\}_n \textit{ such that } \lim_{i\to\infty} a_{n_i} = c\}.$$

(a) (10 points) Prove that if $a_n \le B$ for all n, then $\limsup_n a_n \le B$.

(b) (10 points) Find $\limsup_n a_n$ for

$$a_n = \frac{2n - n^2 - (-1)^n n^2}{n+4}$$

and prove your answer directly from the definition above.

4. (10 points) A function $f : \mathbb{R} \to \mathbb{R}$ is called *integrable on* $\mathbb{R}$ if it is Riemann integrable on all intervals and the improper integral

$$\int_{-\infty}^{\infty} |f|\, dx$$

is finite. Prove that if f is bounded and integrable, then f^2 is integrable on $\mathbb{R}$.

5. (20 points) Prove that the function

$$f(x) = \begin{cases} \frac{\sin(2x)}{3x}; & x \neq 0 \\ \frac{2}{3}; & x = 0. \end{cases}$$

is uniformly continuous on $\mathbb{R}$.

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6. Let

$$H(x) = \begin{cases} 1; & x \leq 0 \\ 0; & x > 0 \end{cases}$$

$H_n(x) = \frac{1}{n}H(x-n)$, and

$$G_m(x) = \sum_{n=1}^{m} \frac{H_n(x)}{n^2}.$$

(a) (10 points) Prove that $G_m(x)$ is Riemann integrable on all bounded intervals.

(b) (10 points) Prove that $\{G_m(x)\}$ uniformly converges to a Riemann integrable function on all bounded intervals.

7. (10 points) Given a continuously differentiable function $f : \mathbb{R}^2 \to \mathbb{R}^2$ so that for all $\mathbf{x} \in \mathbb{R}^2$, there exists integer $N_{\mathbf{x}} > 0$ and invertible 2×2 matrix $A_{\mathbf{x}}$ so that

$$[Df_{\mathbf{x}}]^{N_{\mathbf{x}}} = A_{\mathbf{x}} \begin{pmatrix} 2 & 1 \\ 5 & 5 \end{pmatrix} A_{\mathbf{x}}^{-1},$$

prove that for every $\mathbf{x} \in \mathbb{R}^2$, there exists an open neighborhood U so that $f : U \to f(U)$ is invertible. (Here, $[Df_{\mathbf{x}}]$ is the matrix representation of the linear map $Df_{\mathbf{x}}$.)