國立成功大學

111學年度碩士班招生考試試題

編　號：36

系　所：數學系應用數學

科　目：高等微積分

日　期：0220

節　次：第 2 節

備　註：不可使用計算機

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Throughout the exam, the Euclidean spaces $\mathbb{R}^n$ *are all equipped with usual Euclidean metric* $d(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\|$.

1. A vector field $\mathbf{V}$ on $\mathbb{R}^3$ is called *conservative* if $\mathbf{V} = \nabla f$ for some differentiable function f.

   (a) (5 points) Prove that if a smooth vector field $\mathbf{V} = (V_1, V_2, V_3)$ (i.e. V_i's are smooth) is conservative on some domain $U \subset \mathbb{R}^3$, then

   $$curl\ \mathbf{V} = \left(\frac{\partial V_3}{\partial y} - \frac{\partial V_2}{\partial z}, \frac{\partial V_1}{\partial z} - \frac{\partial V_3}{\partial x}, \frac{\partial V_2}{\partial x} - \frac{\partial V_1}{\partial y}\right) = (0, 0, 0)$$

   on U.

   (b) (10 points) Is the converse of (a) true? Prove it or disprove with a counterexample.

2. (15 points) Let $\{K_\alpha\}_{\alpha \in A}$ be a family of compact subsets of a metric space (X, d) such that any finite intersection of K_α's is nonempty (i.e. $\cap_{\alpha \in F} K_\alpha \neq \emptyset$ for all finite subset $F \subset A$), prove that

   $$\bigcap_{\alpha \in A} K_\alpha \neq \emptyset.$$

3. (10 points) Prove, for $s > 1$, that

   $$\sum_{n=1}^{\infty} \frac{1}{n^s} = s \int_1^{\infty} \frac{[x]}{x^{s+1}}\, dx,$$

   where $[x]$ is the greatest integer $\leq x$.

4. (15 points) Let $\{f_n\}_n$ be a sequence of real-valued functions defined on a compact metric space (K, d) so that

   - $f_n \geq f_{n+1} \geq 0\ \forall n$
   - $f_n \to 0$ pointwise.

   Prove that f_n converges to 0 uniformly.

   (*Hint:* Problem 2 can be helpful.)

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5. (a) (5 points) State the *Inverse Function Theorem* for a function $F : \mathbb{R}^n \to \mathbb{R}^n$.

   (b) (10 points) Use the *Rank Theorem* below to prove the Inverse Function Theorem.

   Rank Theorem: Let $F : \mathbb{R}^n \to \mathbb{R}^m$ be a differentiable function so that $DF_{\mathbf{p}}$ has rank k at some $\mathbf{p} \in \mathbb{R}^n$. Then, there exist open neighborhoods U of $\mathbf{p}$ and V of $F(\mathbf{p})$, and C^1 maps

$$\varphi : U \to \varphi(U) \subset \mathbb{R}^n, \quad \psi : V \to \psi(V) \subset \mathbb{R}^m,$$

   both invertible with C^1 inverses, so that

$$\psi \circ F \circ \varphi^{-1}(x_1, \ldots, x_k, x_{k+1}, \ldots, x_n) = (x_1, \ldots, x_k, 0, \ldots, 0).$$

6. (15 points) Let $f(x, y)$ be a function on $\mathbb{R}^2$ such that $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exist everywhere and are both bounded, prove that f is continuous.

7. (15 points) Let $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ be a map so that

$$\|\varphi(\mathbf{x}) - \varphi(\mathbf{y})\| < c\|\mathbf{x} - \mathbf{y}\|$$

   for some $c \in (0, 1)$. Prove that there exists a unique point $\mathbf{x}_0 \in \mathbb{R}^n$ so that

$$\varphi(\mathbf{x}_0) = \mathbf{x}_0.$$