# 國立成功大學

# 112學年度碩士班招生考試試題

編　號：36

系　所：數學系應用數學

科　目：高等微積分

日　期：0207

節　次：第 2 節

備　註：不可使用計算機

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※ 考生請注意：本試題不可使用計算機。　請於答案卷(卡)作答，於本試題紙上作答者，不予計分。

In this exam, $\mathbb{R}^n$ is equipped with the Euclidean norm $\|\cdot\|$.

(1) (20 points) Let $(M,d)$ be a metric space satisfying the property[1]: for each $x \in M$, there exists an open neighborhood $U$ of $x$ such that $U$ is path connected. Prove that if $M$ is connected, then $M$ is path connected.

(2) (15 points) Let $\{a_n\}$ be a sequence of real numbers with $|a_n| < 1$. Suppose that

$$|a_{n+2} - a_{n+1}| < \frac{1}{5}|a_{n+1}^3 - a_n^3|, \quad n \geq 1.$$

Prove that $\{a_n\}$ is convergent in $\mathbb{R}$.

(3) (15 points) Prove or disprove that $\sum_{n=1}^{\infty} \sin^2\left(\frac{\pi}{2n}\right)\cos(nx)$ converges uniformly on $[-\pi,\pi]$.

(4) (20 points) Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation.

(a) (10 points) Prove that there exists $M > 0$ so that $\|T(x)\| \leq M\|x\|$ for any $x \in \mathbb{R}^n$.

(b) (10 points) Define

$$\|T\| = \sup_{x \in S^{n-1}} \|T(x)\|,$$

where $S^{n-1} = \{x \in \mathbb{R}^n : \|x\| = 1\}$. Prove that there exists $x_0 \in S^{n-1}$ so that $\|T\| = \|T(x_0)\|$.

(5) (20 points) Let $B = \{x \in \mathbb{R}^n : \|x\| < 1\}$. Define a function

$$f : B \to \mathbb{R}^n, \quad f(x) = \frac{x}{\sqrt{1-\|x\|^2}}.$$

(a) (10 points) Show that $f$ is differentiable at every point of $B$. Compute the matrix representation of the derivative $Df(x)$ of $f$ at each point $x$ of $B$ relative to the standard basis for $\mathbb{R}^n$.

(b) (10 points) Let $J_f(x) = \det Df(x)$ be the Jacobian of $f$ at $x$. Prove that $J_f(x) \neq 0$ for all $x \in B$.

(6) (10 points) Let $S$ be the surface defined by

$$z = \sqrt{x^2+y^2}, \quad 1 \leq z \leq 3.$$

Evaluate the surface integral $\iint_S \frac{1}{z^2}\,d\sigma$.

[1]Such a metric space is said to be locally path connected.