國立成功大學
114學年度碩士班招生考試試題

編　號：36

系　所：數學系應用數學

科　目：線性代數

日　期：0211

節　次：第 1 節

注　意：1.不可使用計算機
　　　　2.請於答案卷(卡)作答，於
　　　　　試題上作答，不予計分。

LINEAR ALGEBRA

You need to write down your arguments and show your calculations in details in order to get full credit. In case you wish to quote a theorem, please write down the statement(s) clearly and make sure that you check all the hypotheses are fulfilled.

Notation. The following notations will be used.

- $\mathbb{R}$ denotes the set of all real numbers.
- $\mathbb{C}$ denotes the set of all complex numbers.
- For any field F, $\mathrm{Mat}_{m\times n}(F)$ denotes the set of all $m \times n$ matrices whose entries are in F.
- For any field F, $P_n(F)$ denotes the set of all polynomials over F whose degree is *less than* or *equal to* n.
- For a vector space V over F, denote by $V^* = \mathcal{L}(V, F)$ the dual space of V, i.e. the space of all linear maps from V to F.

Problem 1. Let $V = P_2(\mathbb{R})$. Let $\beta := \{1, 1-x, x+x^2\}$ be an ordered basis for V. Consider the linear map $T\colon V \to V$ defined by

$$T(ax^2 + bx + c) = (-11a + 12b + 6c)x^2 + (4a - b - 2c)x + (-24a + 24b + 15c).$$

(a) (10%) Compute the matrix $[T]_\beta$.
(b) (15%) Determine whether or not T is diagonalizable. If so, find an ordered basis γ for V such that $[T]_\gamma$ is diagonal. If not, explain the reason.

Problem 2 (15%). Compute the Jordan canonical form J of the matrix

$$A = \begin{bmatrix} 2 & -4 & 2 & 2 \\ -2 & 0 & 1 & 3 \\ -2 & -2 & 3 & 3 \\ -2 & -6 & 3 & 7 \end{bmatrix}.$$

Also find an invertible matrix $Q \in \mathrm{Mat}_{4\times 4}(\mathbb{C})$ such that $J = Q^{-1}AQ$.

Problem 3. Let V be a finite-dimensional vector space over a field F. For a subset $S \subset V$, we define the annihilator S^0 of S as the subset

$$S^0 := \{f \in V^* \mid f(x) = 0 \text{ for all } x \in S\}.$$

Show that

(a) (5%) S^0 is a subspace of V^*.
(b) (10%) $(W_1 + W_2)^0 = W_1^0 \cap W_2^0$ for subspaces W_1 and W_2.

Problem 4 (10%). Let V and W be a finite-dimensional vector space over a field F and $T\colon V \to W$ be linear. Define

$$T^t\colon W^* \to V^* \text{ by } g \mapsto g \circ T.$$

Show that $\mathrm{Ker}(T^t) = (\mathrm{Im}(T))^0$.

Problem 5 (10%). Determine whether or not the following matrix

$$\begin{bmatrix} 2 & 8 & 12 & -2 \\ 0 & 0 & 1 & 6 \\ 0 & 1 & -5 & -1 \\ 0 & 0 & 0 & 5 \end{bmatrix}$$

is invertible. If so, find its inverse. If not, explain your reasons.

Problem 6 (10%). Define a $n \times n$ matrix $X = (X_{ij}) \in \mathrm{Mat}_{n\times n}(\mathbb{C})$ via $X_{ij} = i \cdot j$. Show that X is diagonalizable. Find all the eigenvalues of X.

Problem 7 (15%). Let $T, S\colon V \to V$ be diagonalizable linear operators on a finite-dimensional vector space V. Show that T and S are simultaneously diagonalizable if and only if $TS = ST$.