國立成功大學
114學年度碩士班招生考試試題

編　號：37

系　所：數學系應用數學

科　目：高等微積分

日　期：0211

節　次：第 2 節

注　意：1.不可使用計算機
　　　　2.請於答案卷(卡)作答，於試題上作答，不予計分。

1. (20 points) Let $\alpha \in \mathbb{R}$ and $f : \mathbb{R}^2 \to \mathbb{R}$ be a function defined by

$$f(x,y) = \begin{cases} |xy|^\alpha \sin \frac{1}{x^2+y^2} & \text{when } (x,y) \neq (0,0) \\ 0 & \text{when } (x,y) = (0,0) \end{cases}.$$

Find all possible α such that f is differentiable at $(0,0)$.

2. (10 points)
Consider the following set:

$$L := \{x \in [0,1] \mid x = 0.0b_1 00b_2 000b_3 \ldots \text{ where } b_i \in \{0,3\} \text{ for all } i \in \mathbb{N}\}.$$

Or equivalently, we can define $x \in L$ when $x = 0.a_1a_2a_3\ldots$ where

$$a_n = \begin{cases} a_n \in \{0,3\} \text{ if } n = \frac{k(k+1)}{2} + k - 1 \text{ for some } k \in \mathbb{N}; \\ 0 \text{ otherwise.} \end{cases}$$

Prove that

(a). (5 points) L is perfect, i.e., L is closed and all elements in L are limit points of L.

(b). (5 points) Find a perfect set by shifting L which contains no rational numbers.

3. (10 points) Let $E \subset \mathbb{R}^2$ be

$$E = \{(0,y) | y \in [-1,1]\} \cup \left\{\left(x, \cos\frac{1}{x}\right) \middle| x \in (0, \pi^{-1})\right\}.$$

(a). (5 points) Prove that E is connected.

(b). (5 points) Prove that E is path-connected, i.e., for any x, $y \in E$, there exists a continuous $f : [0,1] \to E$ satisfies $f(0) = x$ and $f(1) = y$.

4. (20 points) Let $f_n(x) = \sum_{k=1}^{n} k^{-\frac{3}{2}}(kx - [kx])$. Here $[x]$ is the integer part of x.

(a). (6 points) Prove that $\{f_n(x)\}$ converges uniformly to some function f on $\mathbb{R}$.

(b). (7 points) Prove that the discontinuities of f form a countable dense set.

(c). (7 points) Prove that f is Riemann-integrable on every closed interval.

5. (20 points) Let f be a Riemann integrable function defined on $[a,b]$ and $g \in C^0(\mathbb{R})$ be a periodic function with period 1. Prove that

$$\lim_{k\to\infty} \int_a^b f(x)g(kx)dx = \int_a^b f(x)dx \int_0^1 g(x)dx.$$

6. (20 points) Let $u \in C_0^3(\mathbb{R}^3)$ and $\Delta u = f$. Prove that

$$\sum_{j=0}^{2} \int |\nabla^j u|^2 \le 7\Big(\sum_{j=0}^{1} \int |\nabla^j f|^2 + \int |u|^2\Big).$$