國立成功大學

115學年度碩士班招生考試試題

編　號：34

系　所：數學系應用數學

科　目：線性代數

日　期：0203

節　次：第 2 節

注　意：1.不可使用計算機
2.請於答案卷(卡)作答，於試題上作答，不予計分。

Department of Mathematics, National Cheng Kung University

Linear Algebra

Answer all questions. All sub-parts are weighted as indicated. Show all your work and properly justify your answers to earn full credit.

Notation: $\mathbb{R}$: the field of real numbers, $\mathbb{C}$: the field of complex numbers.
$M_n(\mathbb{C})$: the set of $n \times n$ matrices with entries in $\mathbb{C}$, similarly for $M_n(\mathbb{R})$.
$\mathcal{L}(V, W)$: the set of all linear maps from V to W, where V, W are vector spaces.
A^*: the adjoint (or conjugate transpose) of A, where A is a matrix.

Problem 1: (10pts). Consider the following system of linear equations.

$$\begin{cases} x_1 - 3x_2 + 2x_3 - x_4 + 2x_5 = 0 \\ 3x_1 - 9x_2 + 7x_3 - x_4 + 3x_5 = 1 \\ 2x_1 - 6x_2 + 7x_3 + 4x_4 - 5x_5 = 3 \end{cases}$$

Find the solution set S of this linear system, including a particular solution and a basis for its homogeneous solution set S_H.

Problem 2: (15pts). Consider $C[0,1]$, the vector space of real-valued continuous functions on $[0,1]$ equipped with the following standard inner product

$$\langle f, g \rangle = \int_0^1 f(t)g(t)dt.$$

Let W be the subspace of $C[0,1]$ spanned by $\{1, \sqrt{t}, t\}$. Find an **orthogonal** basis for W.

Problem 3: (15pts). Let $T : \mathbb{R}^4 \to \mathbb{R}^4$ be the linear map defined by

$$T(a, b, c, d) = (2a - b - c, -a + 3b + 2c - d, -b + c, a - c + 3d)$$

Find the minimal polynomial of T. To earn full credit, you need to explain why your answer is minimal.

Problem 4: (20pts). Let $A \in M_n(\mathbb{C})$ be a matrix satisfying $A^*A = AA^*$, and let $v \in \mathbb{C}^n$ be a column vector. Show that if $A^m v = 0$ for some integer $m \geq 2$, then $Av = 0$.

Problem 5: . Consider the special orthogonal group

$$SO_n(\mathbb{R}) = \{B \in M_n(\mathbb{R}) \mid \det B = 1 \text{ and } ||Bv|| = ||v||, \ \forall v \in \mathbb{R}^n\},$$

where $\mathbb{R}^n$ is equipped with the usual inner product and norm.

(a.) (10pts) Prove that for any $A \in SO_3(\mathbb{R})$, there exists some non-zero $v \in \mathbb{R}^3$ such that $Av = v$.

(b.) (10pts) Show that the statement above is NOT true for $SO_2(\mathbb{R})$, by providing an explicit counterexample.

Problem 6: (20pts). Let $A, B \in M_3(\mathbb{C})$. Suppose that $B^2 \neq 0$ and $B^3 = 0$. Prove that if $AB = BA$, then $A = f(B)$ for some polynomial $f(x) \in \mathbb{C}[x]$ of degree 2.