國立成功大學
115學年度碩士班招生考試試題

編　號：33

系　所：數學系應用數學

科　目：高等微積分

日　期：0203

節　次：第 1 節

注　意：1.不可使用計算機
2.請於答案卷(卡)作答，於試題上作答，不予計分。

Department of Mathematics, National Cheng Kung University
Graduate Entrance Exam: Advanced Calculus

Jan. 2026

1. *(15 points)* Let $f(x) = \sqrt{x}$ be defined on $[0, 1]$ and $\varepsilon > 0$. Find $\delta > 0$ such that
$$|f(x) - f(y)| < \varepsilon$$
whenever $|x - y| < \delta$.

2. *(20 points)* Let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}$ with $|a_n| > 0$ for all $n \in \mathbb{N}$.

   (a). *(10 points)* Prove that
$$\limsup_{n\to\infty} \sqrt[n]{|a_n|} \le \limsup_{n\to\infty} \frac{|a_{n+1}|}{|a_n|}.$$

   (B). *(10 points)* Give an example which has these two values in the inequality non-equal.

3. *(15 points)* Let X be an open, path-connected set in $\mathbb{R}^2$. Prove that for any differentiable function $f : X \to \mathbb{R}$ satisfying $\nabla f(x) = 0$ for all $x \in X$, f is a constant function.

4. *(15 points)* For any differentiable function f defined on $[a, b]$, we define
$$\|f\| := \sup_{x\in[a,b]} \left(|f(x)| + |f'(x)|\right).$$
Prove that for any sequence $\{f_n\}_{n\in\mathbb{N}}$ which satisfies $\|f_n\| \le 1$, there exists a subsequence converging uniformly on $[a, b]$ to a continuous function.

5. *(20 points)* Let $P \in \mathbb{N}$ be a fixed integer. For each $n \in \mathbb{N}$, we define $h_n : [0, 1] \to \mathbb{R}$ by
$$h_n(x) = \begin{cases} 1 & \text{when } P^n x \in \mathbb{Z} \\ 0 & \text{otherwise .} \end{cases}$$
Prove that

   (a). *(5 points)* The series $\sum \frac{h_n(x)}{K^n}$ converges uniformly when $K > P$.

   (b). *(10 points)* When $K > P$, prove that $\sum \frac{h_n(x)}{K^n}$ is Riemann-Stieltjes integrable.

   (c). *(5 points)* Compute
$$\int_0^1 \sum \frac{h_n(x)}{K^n} dx.$$

6. *(15 points)* Let $U \in \mathbb{R}^3$ be an open set. Recall that $C^2_{cpt}(U)$ is the collection of second order differentiable functions $f : U \to \mathbb{R}$ with

$$supp(f) := \overline{\{x|f(x) \neq 0\}}$$

compact. For any $f \in C^2_{cpt}(U)$, we define

$$\triangle f := (\partial_x^2 + \partial_y^2 + \partial_z^2)f.$$

Prove that the equation

$$\triangle f = e^x f$$

has no solution in $C^2_{cpt}(U)$.