注意事項：

(1) 本試題共分 PART I (Linear Algebra), PART II (Probability) 及 PART III (Numerical Analysis) 三部分。每部分 50 分.

(2) 請選兩部分作答(請將題號標示清楚).

PART I (Linear Algebra)

(I). Let $V = \mathbb{R}^2$ and define $f_1, f_2 \in V^*$, the dual space of V, by $f_1(x,y) = x-2y$ and $f_2(x,y) = x+y$

(i) Prove that $\{f_1, f_2\}$ is a basis for V^*. (4%)

(ii) Find a basis for V for which it is the dual. (8%)

(II). Given $A = \begin{pmatrix} 4 & -2 & 1 \\ -2 & 1 & 2 \\ 1 & 2 & 4 \end{pmatrix}$.

(i) Find all the eigenvalues of A. (6%)

(ii) Determine all the eigenspaces corresponding to the eigenvalues in (i). (6%)

(iii) Find a matrix Q such that $Q^t A Q$ is a diagonal matrix (Q^t is the transpose of Q). (10%)

(iv) Compute A^{20}. (6%)

(III) Let V be the vector space of functions that are linear combinations of e^x, xe^x, x^2e^x and e^{2x}. Define $T: V \longrightarrow V$ via $T(f) = \frac{d}{dx}(f)$. Find a Jordan canonical form for T. (10%)

Part I (Probability Theory)

1. Let X_1 , X_2 be independent random variables with Poisson distribution $P(\lambda_1)$ and $P(\lambda_2)$.

(a) Show that X_1+X_2 has the Poisson distribution $P(\lambda_1+\lambda_2)$. (7%)

(b) Find the conditional probability $P(X_1=k|X_1+X_2=n)$. (7%)

2. Let the random variable θ be distributed uniformly $U(-\pi/2,\pi/2)$ with p.d.f.

$$f(\theta)=\begin{cases} 1/\pi & -\pi/2\leq\theta\leq\pi/2, \\ 0 & \text{otherwise.} \end{cases}$$

(a) Find the p.d.f. of the variable $Y=3\cos\theta$. (6%)

(b) Find $E(Y)=?$ (6%)

3. Let X,Y be two random variables and let Φ_X denote the characteristic function of random variable X.

(a) Prove or disprove: If $E(XY)=E(X)\cdot E(Y)$ then X, Y are independent. (please give the reason) (6%)

(b) Prove or disprove: If $\Phi_{X+Y}(t)=\Phi_X(t).\Phi_Y(t)$, for every t, then X,Y are independent. (please give the reason) (6%)

4. If $X_1,\ldots X_n\ldots$ are i.i.d random variables with $P(X_i=0)=1/6$, $P(X_i=1)=2/6$, $P(X_i=2)=3/6$, let $S_n=X_1+\ldots+X_n$, for $n\geq1$, $S_0=0$; and let $N(t)=$ number of positive integer n for which $S_n\leq t$ (i.e. $N(t)=n$ iff $S_n\leq t$ and $S_{n+1}>t$).

(a) Find the probability $P(N(1)=4)=?$ (6%)

(b) Find the expection $E(N(1))=?$ (6%)

PART III (Numerical Analysis) (50%)

(A) In the iterative methods to solve the problem $AX=b$, $A\in\mathbb{R}^{n\times n}$, We splite

$A=D-C$ for Jacobi process:

$$X_i^{(k+1)}=\frac{-\sum_{j\neq i}a_{ij}x_j^{(k)}+b_i}{a_{ii}},\quad 1\le i\le n,\quad k=0,1,2,\cdots,$$

and splite $A=D(L+I+U)$ for Gauss-Seidel process:

$$X_i^{(k+1)}=\frac{-\sum_{j=1}^{i-1}a_{ij}x_j^{(k+1)}-\sum_{j=i+1}^{n}a_{ij}x_j^{(k)}+b_i}{a_{ii}}.$$

Let $X^{(0)}$ be arbitrary and let $X^{(k+1)}=BX^{(k)}+R$, $B\in\mathbb{R}^{n\times n}$. Let v be a norm on $\mathbb{R}^n$ and let $\|B\|_v=\max_{v(x)=1}v(BX)$.

(1) If λ is the eigenvalue of B, then $|\lambda|\le\|B\|_\infty$. (5%)

(2) If $\|B\|_\infty<1$, then $\lim_{k\to\infty}X^{(k)}=X$. (10%)

(3) Find B in Jacobi iteration and in Gauss-Seidel iteration. (5%)

(4) Discuss the convergence of the Jacobi and Gauss-Seidel process for the solution of $AX=b$ when $A=\begin{pmatrix}1&-2&2\\-1&1&-1\\-2&-2&1\end{pmatrix}$ and when $A=\begin{pmatrix}1&1&-2\\-1&1&-1\\-1&2&1\end{pmatrix}$. (10%)

(B)

(1) Use three steps of the second order Taylor series method with $h=0.5$ to compute an approximation to the solution $y(x)$ at $x=2.5$ for the differential equation $y'(x)=y^2+2x$, $y(1)=1$ (20%)