1. a) Define a sufficient statistic. a.(5%)

b) State the Fisher - Neyman factorization criterion for a statistic to be sufficient. b.(5%)

c) Let $\varphi(X)$ be a test function for a hypothesis testing problem involving $\{P_\theta : \theta \in \Omega\}$ and let $t(X)$ be a sufficient statistic. Prove that $E_\theta[\varphi(X)|t]$ is a test function having the same power function as $\varphi(X)$. c.(6%)

2. Let $X_1, \cdots, X_n$ be independently and identically distributed random variables with a common distribution function F which is absolutely continuous, and let $X_{(1)} \le X_{(2)} \le \cdots \le X_{(n)}$ be the corresponding order statistics of $X_1, \cdots, X_n$. Consider the parameter $\theta = P\{X_1 + X_2 > 0\}$ and statistic $T_n = \frac{1}{\binom{n}{2}} \sum_{i<j}^{n} \sum^{n} \psi(X_i + X_j)$, where $\psi(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x \le 0 \end{cases}$.

(a) Prove that the statistic $t_n = (X_{(1)}, \cdots, X_{(n)})$ is sufficient for the family of absolutely continuous distributions. (Indeed, it is complete sufficient) a.(7%)

(b) Prove that T_n is an unbiased estimator of θ. b.(5%)

(c) Using the Lehmann - Scheffé theorem, show that T_n is the U.M.V.U estimator of θ. c.(8%)

3. Let $X_1, \cdots, X_n$ be a random sample from $N(\theta, 1)$. Consider the problem of estimating θ with a squared error loss function. If the prior distribution of θ is $N(\mu_0, 1)$,

(a) Show that the Bayes estimator of θ is $\frac{\mu_0 + \sum_{i=1}^{n} X_i}{n+1}$. a.(8%)

(b) Find the corresponding Bayes risk (or, average risk). b.(5%)

(c) Is it also a minimax estimator of θ? Explain. c.(7%)

(d) What would be the Bayes estimator if the loss function is the absolute error i.e. $L(\theta, a) = |\theta - a|$ instead of the squared error? Explain. d.(6%)

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4. (a) Let $X_1, \cdots, X_n$ be a random sample from $N(\mu, \sigma^2)$, where both μ and σ^2 are unknown. Find the level α likelihood ratio test for $H_0 : \mu = \mu_0$ against $H_1 : \mu \neq \mu_0$ a.(8%)

(b) The following data shows the difference in sleep gained using the drugs A and B on 10 patient. Let d_i denote the ith difference.

patient i	1	2	3	4	5	6	7	8	9	10
A	+0.7	-1.6	-0.2	0.1	-0.1	+4.4	+3.7	+1.6	0.0	+3.4
B	+1.9	+0.8	+1.1	-1.2	-0.1	+3.4	+5.5	+0.8	+4.6	+2.0
d_i	+1.2	+2.4	+1.3	-1.3	0.0	-1.0	+1.8	-0.8	+4.6	-1.4

Computations give $\sum_{i=1}^{10} d_i = 6.8$, $\sum_{i=1}^{10} d_i^2 = 38.6$. Use the statistic in part (a) to test H_0: no difference between A and B under the level $\alpha = 0.01$. b.(7%)

(c) Use the data of (b) to find a level 99% confidence interval. c.(7%)

5. (a) Let $U_1, \cdots, U_n$ be a random sample from a uniform distribution function on $[0, 1]$, and let $U_{(1)} = \min\{U_1, \cdots, U_n\}$, $U_{(n)} = \max\{U_1, \cdots, U_n\}$. Show that the distribution of the sample range $R = U_{(n)} - U_{(1)}$ is a Beta distribution $\beta(n-1; 2)$. a.(8%)

(b) Let $X_1, \cdots, X_n$ be a random sample from a continuous distribution function F, and let $X_{(1)} = \min\{X_1, \cdots, X_n\}$, $X_{(n)} = \max\{X_1, \cdots, X_n\}$. Using the result in part (a), discuss how to find a sample size n such that at least 99% of certain population, with probability 0.95, will lie between the smallest and largest sample observations. That is to determine n so that $0.95 = P\{F(X_{(n)}) - F(X_{(1)}) \geq 0.99\}$. b.(8%)

Note: $t_9(0.995) = 3.250$ $\quad t_9(0.99) = 2.821$

$t_{10}(0.995) = 3.169$ $\quad t_{10}(0.99) = 2.764$

where $t_k(1-\alpha)$ is the $1-\alpha$ quantile of the Student t-distribution with k degree of freedom.