每一部分至少作一題，共作五題，每題 20分.

PART I.

1. (a) Find the radius of convergence of the power series $\sum_{n=0}^{+\infty} x^{2n}$ and its sum function $f(x)$. (5%)

   (b) Use (a) to represent $F(x) = \ln\sqrt{\frac{1+x}{1-x}}$ as its Maclaurin series. (8%)

   (c) Use (b) to approximate $F(-\frac{1}{3}) = -\frac{1}{2}\ln 2$ such that the absolute error less than 0.005. (7%)

2. Let $f(x) = \tan^{-1}\frac{x-1}{x+1}$:

   (a) Find $\lim_{x\to-\infty} f(x)$; $\lim_{x\to+\infty} f(x)$; $\lim_{x\to-1^-} f(x)$ and $\lim_{x\to-1^+} f(x)$. (8%)

   (b) Verify that $f'(x) = \frac{1}{1+x^2}$. Can we say that f is increaing and f is different from $\tan^{-1}x$ by a constant ? why ? (12%)

三. Evaluate the following improper integrals. (20%)

   (a) $\int_0^{+\infty} e^{-x^2}dx$; (b) $\int_1^{+\infty} \frac{(-1)^{[x]}}{[x]}dx$.

PART II.

4. Let $\{r_1, r_2, \cdots, r_n, \cdots\} = [0,1]\cap\mathbb{Q}$, $\forall n\in\mathbb{N}$, $f_n : [0,1]\to\mathbb{R}$

$$f_n(x) = \begin{cases} \sinh x, & \text{if } x\in\{r_1, r_2, \cdots, r_n\} \\ 0, & \text{otherwise.} \end{cases}$$

   (a) Find the limit function $f(x)$. (5%)

   (b) Does the equality $\int_0^1 f(x)dx = \lim_{n\to+\infty}\int_0^1 f_n(x)dx$ hold as Riemann integral ? as Lebesgue integral ? (10%)

   (c) Does $\{f_n\}\to f$ converge uniformly on $[0,1]$? why ? (5%)

5. Let $G(x,y,z)=\int_x^y \frac{\sin zt}{t}\,dt$, and $F(x)=G(x,\sqrt{x},x^2)$, $x>0$. Find $F'(x)=?$ (20%)

6. Let f and g be continuous functions on $\mathbb{R}$ and $f(x)=g(x)$ when $x\in D$, where D is dense in $\mathbb{R}$. Prove that

$$f(x)=g(x), \text{ for every } x\in\mathbb{R}. \qquad (20\%)$$

Part III

7. For $i,j=1,2$, let $a_{ij}(t)$ and $b_{ij}(t)$ be continuous functions over $I=(\alpha,\beta)$ and let $A(t)=(a_{ij}(t))$ and $B(t)=(b_{ij}(t))$. For $k=1,2,3,4$, let $\vec{X}_k=\begin{pmatrix}x_{k1}\\x_{k2}\end{pmatrix}$ be any four solutions of the differential equation

$$\vec{X}''=A(t)\vec{X}'+B(t)\vec{X}, \quad t\in I.$$

Denote $W(\vec{X}_1,\vec{X}_2,\vec{X}_3,\vec{X}_4)$

$$=\det\begin{pmatrix} x_{11} & x_{21} & x_{31} & x_{41}\\ x_{12} & x_{22} & x_{32} & x_{42}\\ x_{11}' & x_{21}' & x_{31}' & x_{41}'\\ x_{12}' & x_{22}' & x_{32}' & x_{42}'\end{pmatrix}.$$

(a) Show that $\frac{dW}{dt}=(a_{11}+a_{22})W$. (15%)

(b) Show that either W is identically zero or else never vanishes over I. (5%)

8. (1) Construct the Green's function for the differential equation $y''+y=0$ with the boundary conditions $y(0)=y(1)=0$. (15%)

(2) Find a formula for a particular solution of $y''+y=f(t)$, $y(0)=y(1)=0$, for $0<t<1$. (5%)