國立成功大學 77 學年度應用數學研究所/研究所入學考試(基礎數學(B)試題) 共 3 頁 第 1 頁

注意事項：

(1) 本試題共分 PART I (Linear Algebra), Part II (Probability) 及 Part III (Numerical Analysis) 三部分。每部分 50分

(2) 請選兩部分作答 (請將題號標示清楚)

Part I (Linear Algebra)

1. Let $\mathcal{T}=\{(x,y,z,w)\mid x+y+z+w=0\}$ and $\mathcal{S}=\{(x,y,z,w)\mid x-y+z-w=0\}$ be two subspaces of $\mathbb{R}$. Find a basis for $\mathcal{T}\cap\mathcal{S}$ and write down $\dim \mathcal{T}\cap\mathcal{S}$. (10%)

2. Find a unitary matrix P that diagonalizes $A=\begin{pmatrix} 2 & 1+i \\ 1-i & 3 \end{pmatrix}$ (12%)

3. Given $A=\begin{pmatrix} 2 & 1 & -6 & -6 \\ 0 & 2 & 0 & 0 \\ -3 & -1 & 5 & 6 \\ 3 & 1 & -6 & -7 \end{pmatrix}$.

(a) Find the minimal polynomial of A.

(b) Find the Jordan canonical form of A. (12%)

4. Let V and W be two finite-dimensional inner product spaces over the same field and $\{v_1, v_2, \ldots, v_n\}$ an orthonormal basis for V. Prove that $f: V\to W$ is an inner product isomorphism if and only if $\{f(v_1), f(v_2), \ldots, f(v_n)\}$ is an orthonormal basis for W.
(Hint: $f: V\to W$ is called an inner product isomorphism if it is a vector space isomorphism that preserves inner product, i.e. $\langle f(x), f(y)\rangle = \langle x, y\rangle$ $\forall x, y\in V$.) (16%)

PART II [Probability]

1. 設 $A_1, A_2, \cdots, A_n, \cdots$ 為一獨立事件序列，試問：

(a) A_1 與 $\bigcup_{j=2}^{n} A_j$ 是否為獨立？為何？ (8分)

(b) A_1 與 $\bigcup_{j=2}^{+\infty} A_j$ 是否為獨立？為何？ (8分)

2. 設 X 與 Y 為二隨機變數且 $E[X^2]<+\infty$, $E[Y^2]<+\infty$.

(a) 試証：$(E[XY])^2 \le E[X^2]\cdot E[Y^2]$； (10分)

(b) 利用 (a) 証明：$-1 \le \rho(X,Y) \le 1$，其中 $\rho(X,Y)$ 為 X 與 Y 之相關係數. (8分)

3. 設 X_n 之機率密度函數(p.d.f.)為

$$f_n: \mathbb{R} \to [0,1] \text{ 而 } f_n(x) = \begin{cases} 1/n, & \text{若 } x \in A_n, \\ 0, & \text{若 } x \in \mathbb{R} \setminus A_n, \end{cases}$$

其中 $A_n = \{1/n, 2/n, \cdots, n/n\}$, $\forall n \in \mathbb{N}$.

又設 $\{X_n\}_n$ 分配收斂於 X，即 $X_n \xrightarrow{d} X$.

(a) 利用分配收斂之定義，試求 X 之分配； (8分)

(b) 利用 Lévy 連續性定理(continuity theorem)，即以特徵函數之極限方法，試求 X 之分配. (8分)

代數

PART III (数值分析)

I. Let A be an $n\times n$ matrix, $\vec{x}_t$ be the true solution and $\vec{x}_c$ be a computed solution of $A\vec{x}=\vec{b}$.

(a) Describe the procedure of the "iterative improvement" for refining $\vec{x}_c$ to $\vec{x}_c'$. (5%)

(b) Under what condition one can expect that $\vec{x}_c'$ might be a relatively better approximation to $\vec{x}_t$ than is $\vec{x}_c$. (5%)

II (a) Find the interpolating polynomial $P_2(x)$ of degree ≤ 2 for the function $f(x)=1/(x-2)$, $x\in[-1,1]$, in both Lagrange form and Newton's forward formula at $-1, 0, +1$. (10%)

(b) Find the best least-squares polynomial approximation of degree ≤ 2 for $f(x)=1/(x-2)$, $x\in[-1,1]$, if the inner product is given by $\langle f,g\rangle=\int_{-1}^{1} f(x)g(x)\,dx$.
[Hint: $\{1, x, x^2-\frac{1}{3}\}$ is an orthogonal set.] (5%)

III. Approximate $\int_0^{\pi}\sin x\,dx$ with an absolute error of at most 0.0005, using Simpson's and Trapezoidal rule. (10%)

IV. Approximate the solution of the initial-value problem

$$y'=f(x,y)=-y+x+1,\quad 0\le x\le 1,\quad y(0)=1,$$

(a) by Euler's method, and (5%)

(b) by the Runge-Kutta method of order four with $N=5$, $h=0.2$ and $x_i=0.2i$. (5%)

(c) compare the accuracy for the above two methods. (5%)