國立成功大學七十八學年度應用數學研究所考試(基礎數學(B)試題)

(甲乙組)

注意事項：

本試題共分 PART I (Linear Algebra) 及 PART II (probability) 兩部分。作答時請將題號標示清楚。

PART I (Linear Algebra)

1. Let V be a finite-dimensional vector space over the field F and let $T\ (\neq 0)$ be a linear operator on V.
   Let $W_i = \{ v \in V \mid T^i(v) = 0 \}$, $(i = 0, 1, 2, \cdots)$
   If $v_1, \cdots v_n \in V$ are linearly independent over F such that $\langle v_1, \cdots, v_n \rangle \cap W_i = \{0\}$ for $i \geq 1$.
   where $\langle v_1, \cdots, v_n \rangle$ denote the subspace of V spanned by $\{v_1, \cdots v_n\}$.
   Show that (1) $T(v_1), \cdots, T(v_n)$ are linearly independent over F. (7%)
   (2) $\langle T(v_1), \cdots, T(v_n) \rangle \cap W_{i-1} = \{0\}$, $(i \geq 1)$. (12%).

2. Given $A = \begin{pmatrix} 5 & -4 & 4 \\ 12 & -11 & 12 \\ 4 & -4 & 5 \end{pmatrix}$.

   Find a matrix P such that $P^{-1}AP$ is diagonal, or else show that A is not diagonalizable. (12%).

3. (1) Let A and B be $n \times n$ matrices with entries in a field F. Show that $tr(AB) = tr(BA)$. where $tr(A)$ denote the trace of A. (7%).
   (2) Let $A = (a_{ij})$ be a complex $n \times n$ matrix with eigenvalues $\lambda_1, \cdots \lambda_n$. If A is unitarily equivalent to a diagonal matrix, Show that $$\sum_{i,j=1}^{n} |a_{ij}|^2 = \sum_{i=1}^{n} |\lambda_i|^2 .$$ (12%)

國立成功大學 78學年度應数所 考試(基礎数學(B)試題)

Part II (Probability). (每題 10分)

1. 設 X 為正整数值之隨機变数且 $EX<\infty$. 證明

$$EX = \sum_{j=1}^{\infty} P\{X \geq j\}.$$

2. 設經過某座橋的車輛数目在時間區間 $[t_1, t_2]$ $(0<t_1<t_2)$ 為服從 Poisson 分配 $P(\lambda(t_2-t_1))$. 令 T 表兩輛連續經過車輛間之時間距離. 求 T 之分配函数.

3. 令 $\hat{\sigma} = \sqrt{\frac{1}{n}\sum_{i=1}^{n}(X_i-\bar{X})^2}$, $\tilde{\sigma} = \frac{1}{n}\sum_{i=1}^{n}|X_i-\bar{X}|$ 其中 $X_1, \cdots, X_n$ 為服從 $N(\mu, \sigma^2)$ 之樣本.

(a) 求 $E\hat{\sigma}$ (b) 求 $E\tilde{\sigma}$ (Hint: 先求 Gamma 分配之 moment)

4. 設 X_1, X_2 為獨立且服從對数分配 $(\lambda=1)$ 之隨機變数.

(a) 求 X_1/X_2 之 p.d.f (b) 求 $X_1^2+X_2^2$ 之 p.d.f.

5. (a) 敘述強大数法則 (The strong law of large numbers).

(b) 敘述中央極限定理 (The central limit theorem).