注意事項：

本試題共分 PART I (Linear Algebra) 及 PART II (Numerical Analysis) 兩部分. 作答時請將題號標示清楚.

PART I (Linear Algebra)

1. Let V be a finite-dimensional vector space over the field F and let $T\ (\neq 0)$ be a linear operator on V.
   Let $W_i = \{ v \in V \mid T^i(v) = 0 \}$, $(i = 0, 1, 2, \cdots)$.
   If $v_1, \cdots v_n \in V$ are linearly independent over F such that
   $\langle v_1, \cdots, v_n \rangle \cap W_i = \{0\}$ for $i \geq 1$,
   where $\langle v_1, \cdots, v_n \rangle$ denote the subspace of V spanned by $\{v_1, \cdots v_n\}$.
   Show that (1) $T(v_1), \cdots T(v_n)$ are linearly independent over F (7%)
   (2) $\langle T(v_1), \cdots T(v_n) \rangle \cap W_{i-1} = \{0\}$, $(i \geq 1)$. (12%)

2. Given $A = \begin{pmatrix} 5 & -4 & 4 \\ 12 & -11 & 12 \\ 4 & -4 & 5 \end{pmatrix}$

   Find a matrix P such that $P^{-1}AP$ is diagonal, or else show that A is not diagonalizable. (12%)

3. (1) Let A and B be $n \times n$ matrices with entries in a field F. Show that $tr(AB) = tr(BA)$. where $tr(A)$ denote the trace of A. (7%)
   (2) Let $A = (a_{ij})$ be a complex $n \times n$ matrix with eigenvalues $\lambda_1, \cdots \lambda_n$. If A is unitarily equivalent to a diagonal matrix
   Show that $\sum_{i,j=1}^{n} |a_{ij}|^2 = \sum_{i=1}^{n} |\lambda_i|^2$ (12%).

PART II (Numerical Analysis) (50%):

1. (1) Derive the secant method and the Newton-Raphson method to compute an approximation to a root of the equation $f(x) = 0$. 8%

   (2) Prove that the secant method and the Newton-Raphson method have the rate of convergence (or are of order) $(1 + \sqrt{5})/2$ and 2, respectively. 10%

   (3) Give a good strategy plan in approximating a root of the equation

   $$\cos x - x(\exp x) = 0,$$

   such that $|f(a)| < 0.000001$, where a is the approximation to the root. Carry out your plan numerically. 12%

2. (1) The following are supposed to be the values of a polynomial $p(x)$ of degree $n < 10$. Locate any errors in the following table:

x	0	0.2	0.4	0.6	0.8	1.0
p(x)	1.500	1.590	1.392	0.954	0.324	-0.540

x	1.2	1.4	1.6	1.8	2.0
p(x)	-1.320	-2.238	-3.156	-4.156	-4.800

10%

   (2) Find the order of the following method in computing an approximate solution to a differential equation:

   $$y_{n+1} = y_n + 3k_1/4 + k_2/4,$$
   $$k_1 = hf(t_n + h/3,\ y_n + k_1/3),$$
   $$k_2 = hf(t_n + h,\ y_n + k_1).$$

   Determine the interval of absolute stability when the above method is applied to the test equation $y' = cy$, $c < 0$. 10%