共五題

1. Solve the following problems:

(a) $y'(t) + 2y(t) - 8\int_0^t y(s)ds = -2H(t)$, $y(0)=-5$, where $H(t)=\begin{cases}1 & t>0\\ 0 & t<0\end{cases}$. (10%)

(b) $y'' + (y')^2 + y = 0$, $y(0) = \frac{1}{2}$, $y'(0) = 0$. (10%)

2. Construct the Green's function of the equation

$$y'' + 2y' + 2y = 0$$

with the boundary conditions $y(0) = y(\frac{1}{2}\pi) = 0$, and then find the solution of the equation $y''+2y'+2y=f$ with the above boundary conditions. (20%)

3. Find the equation of the family of curves which have the property that the tangent at any point P determines with the coordinate axes a triangle whose area is equal to twice the area of the rectangle determined by P and the coordinate axes. (20%)

4. Let L be a differential operator defined by $Lu = (pu')' + qu$, where p is a continuously differentiable function and q a continuous function defined on (a,b) and the domain of L is $C^2(a,b)$, the space of twice continuously differential functions defined on (a,b).

(a If u_1, u_2 are solutions of Lu=0, show that $p(u_1u_2'-u_2u_1')$ is a constant. (5%)

(b Suppose that $p \neq 0$ in (a,b). If the two solutions u_1 and u_2 of Lu=0 vanish at some point in (a,b), show that u_1 and u_2 are dependent in (a,b). (5%)

5. Consider the linear system $\vec{x}'(t) = A\vec{x}(t)$, where $\vec{x}(t)=(x_1(t),\ldots,x_n(t))$, x_j's are continuously differentiable functions defined on R^1 and A is a n by n constant matrix.

(a Show that there is a unique n by n matrix function G(t) such that G(0)=I, the identity matrix on R^n, and $G'(t)=AG(t)$. (4%)

(b Show that $G(t+s)=G(t)G(s)$, $AG(t)=G(t)A$, $G(-t)=G(t)^{-1}$ and $G(t-s)=G(t)G(s)^{-1}$. (6%)

(c When n=2 and $A=\begin{pmatrix}1 & 1\\ 3 & -1\end{pmatrix}$, find G(t). (10%)

(d) Let A be the matrix as given in (c) of Problem 5. Denote by W the class of continuous R^2-valued functions $\vec{x}(t) = (x_1(t), x_2(t))$ and denote by V the class of those functions in W such that $\vec{x} = (x_1, x_2)$ is continuously differentiable and $\vec{x}(0) = 0$. Define the operator L by

$$L\vec{x} = \vec{x}' - A\vec{x}.$$

Show that L is injective from V onto W and then find the inverse operator of L. (10%)