1. (a) State the definition of a minimal sufficient statistic. (5%)
   (b) State the definition of a complete statistic. (5%)
   (c) Let $X_1, \ldots, X_n$ be a random sample from Unif(α, β). Find a complete sufficient statistic if any, or a minimal sufficient statistic for (α, β) in cases: (i) $\alpha = 0$, (ii) $-\alpha = \beta$. (10%)

2. (a) State the Basu Theorem related to sufficiency, completeness, and stochastic independence. (10%)
   (b) Let $X_1, \ldots, X_n$ be a random sample from a normal distribution $N(\mu, \sigma^2)$. Prove that

$$\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i \quad \text{and} \quad S^2 = \frac{1}{n}\sum_{i=1}^{n} (X_i - \bar{X})$$

   are independent. (10%)

3. Let $X_1, \ldots, X_n$ be a random sample from the gamma distribution with α known and $\beta = \theta > 0$ unknown,

$$f(x) = \frac{1}{\Gamma(\alpha)\beta^{\alpha}} x^{\alpha-1} e^{-x/\beta}, \quad x > 0.$$

   (a) Find the Cramér-Rao lower bound for unbiased estimators of θ. (10%)
   (b) Find the UMVUE of θ if such exists. (10%)

4. Let $X_1, \ldots, X_n$ be a random sample from a normal distribution $N(0, \sigma^2)$.
   (a) Derive the UMP size α test of $H_0 : \sigma = \sigma_0$ against $H_1 : \sigma > \sigma_0$. (10%)
   (b) Express the power function of this test in terms of a chi-square distribution. (10%)

5. Let $X_1, \ldots, X_n$ be a random sample from a Bernoulli distribution,

$$f(x|\theta) = \theta^{x}(1-\theta)^{1-x}, \quad x = 0, 1.$$

   Consider the problem of estimating θ with a "weighted" squared error loss

$$I(t;\theta) = \frac{(t-\theta)^2}{\theta(1-\theta)}.$$

   If the prior distribution of θ is Unif(0, 1),
   (a) find the corresponding Bayes estimator of θ, (10%)
   (b) prove that the Bayes estimator in (a) is also the minimax estimator of θ corresponding the loss function defined. (10%)