Do any five of six problems, each 20%.

1. (a) Graph $y = \frac{\ln x}{x}$, $(x > 0)$, giving its essential shape, local extrema, and asymptotic behavior.
   (b) Prove that if s and t are distinct positive integers such that $s^t = t^s$, then either $s = 2$ and $t = 4$, or $s = 4$ and $t = 2$.

2. (a) Use the mean value theorem or any other method to evaluate $\lim_{x \to \infty} \sin(x + \frac{1}{x}) - \sin x$.
   (b) Suppose a continuous function $f : [0, 1] \to \mathbb{R}$ satisfies that $f(x) = \int_0^x f(t)\,dt$, $\forall x \in [0, 1]$. Determine the range of f.

3. Evaluate the line integral $\int_C xy^2 + 2x^2y\,dy$, where C is the ellipse $x^2 + 2y^2 = 1$ oriented counter-clockwise.

4. Let $F(x) = \sum_{k=1}^{\infty} \frac{\cos(kx)}{k^3}$, $(x \in \mathbb{R})$.
   (a) Compute $\int_0^{2\pi} F(x)\,dx$. Justify the process of your calculation.
   (b) Compute $F'(x)$. Justify the process of your calculation.

5. Let $f : [0, 1] \longrightarrow \mathbb{R} : f(x) = \begin{cases} 0, & \text{if } x \notin \{\frac{1}{n} : n \in \mathbb{N}\} \\ x, & \text{if } x \in \{\frac{1}{n} : n \in \mathbb{N}\} \end{cases}$.
   (a) Is f Riemann integrable? Justify your answer.
   (b) Is f Lebesgue integrable? Justify your answer.

6. Suppose that $\begin{cases} \text{(i) } f \text{ and } \frac{\partial f}{\partial t} \text{ are continuous on } [a, b] \times [c, d], \\ \text{(ii) } h : [c, d] \to [a, b] \text{ is continuously differentiable.} \end{cases}$
   Define $G : [a, b] \times [c, d] \to \mathbb{R} : G(x, t) = \int_a^x f(s, t)\,ds$.
   (a) Compute $\frac{\partial G}{\partial x}$ and $\frac{\partial G}{\partial t}$.
   (b) $\frac{d}{dt} \int_a^{h(t)} f(s, t)\,ds = ?$