PART I (Linear Algebra).

1. Let W be the subspace of $\mathbb{R}^4$ which is spanned by $(1,1,0,-1)$ and $(2,1,0,-3)$.
   (a) Apply the Gram-Schmidt process to find an orthonormal basis of W. (6%)
   (b) Find the orthogonal projection of the vector $(1,2,3,6)$ on W. (6%)

2. Let $A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$.
   (a) Find the characteristic polynomial and minimal polynomial of A. (6%)
   (b) Find an orthogonal matrix P and a diagonal matrix D such that $A = PDP^T$ where P^T denote the transpose of P. (8%)
   (c) Is A positive definite? Explain your answer. (4%)

3. Let V be a finite-dimensional vector space, and T, S be two linear operators on V. $N(T)$ and $R(T)$ denote the null space and range of T respectively.
   (a) Prove that nullity ST = nullity $T + \dim(R(T) \cap N(S))$. (7%)
   (b) If $\operatorname{rank}(T) = \operatorname{rank}(T^2)$, prove that $V = R(T) \oplus N(T)$. (7%)
   (c) Prove that there exists a positive integer k such that $V = R(T^k) \oplus N(T^k)$. (6%)

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