

PART I (Linear Algebra)

1. Let W be the subspace of \mathbb{R}^4 which is spanned by $(1, 1, 0, -1)$ and $(2, 1, 0, -3)$.

(a) Apply the Gram-Schmidt process to find an orthonormal basis of W . (6%)

(b) Find the orthogonal projection of the vector $(1, 2, 3, 6)$ on W . (6%)

2. Let $A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$.

(a) Find the characteristic polynomial and minimal polynomial of A . (6%)

(b) Find an orthogonal matrix P and a diagonal matrix D such that $A = PDP^T$ where P^T denote the transpose of P . (8%)

(c) Is A positive definite? Explain your answer. (4%)

3. Let V be a finite-dimensional vector space, and T, S be two linear operators on V . $N(T)$ and $R(T)$ denote the null space and range of T respectively.

(a) Prove that $\text{nullity } ST = \text{nullity } T + \dim(R(T) \cap N(S))$. (7%)

(b) If $\text{rank}(T) = \text{rank}(T^2)$, prove that $V = R(T) \oplus N(T)$. (7%)

(c) Prove that there exists a positive integer k such that $V = R(T^k) \oplus N(T^k)$. (6%)