1. Find the general solutions for the following equations.

(a) $y'(x) = 1 + \exp(x + y + 1)$. (10%)

(b) $xdy - ydx = (xy)^{\frac{1}{2}}dx$. (10%)

(c) $x^2y'' + 7xy' + 9y = 0$. (10%)

2. Find the solution of the initial value problem

$$y'' + 2y' + 2y = \delta(t-1) , \quad y(0) = 1 , \quad y'(0) = 0. \qquad (10\%)$$

3. Consider the following system:

$$\vec{x}' = \begin{pmatrix} 2 & -1 \\ 1 & 4 \end{pmatrix} \vec{x} , \quad \text{where } \vec{x}=(x_1,x_2)^t.$$

(a) Find a fundamental set of solutions for the system of equations. (15%)

(b) Let $G(t)$ be the fundamental matrix such that $G(0)=I$. Find the value for

$$\int_{R^2} \exp(-|G(1)\vec{x}|^2)\, d\vec{x} ,$$

where $d\vec{x}=dx_1dx_2$. (10%)

4. Consider the equation on an interval $[0,c]$

$$L[y] = y'' + ay' + by = 0$$

where a, b, c are constant. Suppose that $Z(t)$ is the solution satisfying the initial conditions $y(0)=0$, $y'(0)=1$. Define the mapping G by

$$Gf(t) = \int_0^t Z(t-s)f(s)ds$$

for all continuous functions f on $[0,c]$. Show that $L[Gf]=f$. (10%)

5. Let $L[y] = -[py']' + qy$, where $p(x)$ and $q(x)$ are continuous functions on $[0,1]$ and $q(x) \geq 0$, $p(x) > 0$. Let $\mathcal{M}$ denote the class of twice continuously differentiable functions y satisfying the following boundary conditions:

$$ay(0) + y'(0) = 0$$
$$by(1) + y'(1) = 0.$$

(a) Show that $\langle L[u],v \rangle = \langle u,L[v] \rangle$, for all u, v in $\mathcal{M}$, where 034

$\langle u,v \rangle = \int_0^1 u(t)\overline{v(t)}dt$. (10%)

(b) If $a \leq 0$, $b \geq 0$, show that all eigenvalues of L (with domain $\mathcal{M}$) are positive. (10%)

(c) Show that all the eigenvalues of L (with domain $\mathcal{M}$) are real for arbitrary real number a and b.