1 (a) Let $x_n = \int_1^n \frac{\sin(t+1)^2}{t^2} dt$, $n \in \mathbb{N}$.

Is $\{x_n\}$ convergent to a real number ? 10%

(b) Suppose $a > -1$. Evaluate $\lim_{n \to \infty} \frac{1^a + 2^a + \cdots + n^a}{n^{1+a}}$. 10%

2. At time t a particle has position vector

$$\vec{a}(t) = \left(1 + \tan^{-1} t, \ 1 - \ln\sqrt{1+t^2} \right).$$

(a) Find its velocity vector $\vec{v}(t)$ at time t. 5%

(b) Find the total distance traveled from time $t = 0$ to time $t = 1$. 5%

3. Suppose $\mathbb{R}$ is the region in $\mathbb{R}^3$ bounded by $x = 0$, $y = 0$, $z = 0$

and $\sqrt{\frac{x}{a}} + \sqrt{\frac{y}{b}} + \sqrt{\frac{z}{c}} = 1$, where $a > 0$, $b > 0$ and $c > 0$. Find the value

of the triple integral $\iiint_R z \, d(x,y,z)$. 15%

4. Express the iterated integral

$$\int_0^1 dy \int_0^{f(y)} xy \, dx, \quad \text{where } f(y) = \min\{1, \ln\frac{1}{y}\},$$

as an integral over a set $Q \subseteq \mathbb{R}^2$, and then as an iterated integral in the opposite order. Evaluate it. 12%

5. Determine a differentiable function $f : \mathbb{R} \longrightarrow \mathbb{R}$ such that $f(0) = 0$ and

$$f'(\ln x) = \begin{cases} 1, & \text{if } 0 < x \le 1, \\ x, & \text{if } \quad x > 1. \end{cases}$$ 13%

6. Let a, b, c, d be fixed real numbers. Is there a root lying in $(0, 1)$ for the equation $5ax^4 + 4bx^3 + 3cx^2 + 2dx = a + b + c + d$? 15%

7. Determine the real number α so that there is a value c for which

$$\int_0^c \frac{dx}{1+x^\alpha} = \int_c^\infty \frac{dx}{1+x^\alpha}.$$ 15%

035