1. (i) Find all real numbers x for which the power series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n} x^n$ converges absolutely, and all x for which it converges conditionally. 10%

   (ii) Show that $\sum_{n=1}^{\infty} \frac{\sin \frac{1}{n}}{n}$ converges. 10%

2. Suppose $f : [a,b] \longrightarrow \mathbb{R}$ has the property that if $a \le x < y \le b$ and u is between $f(x)$ and $f(y)$ then there is some z with $x < z < y$ such that $f(z) = u$.

   (i) Show that if f is monotone, then f is continuous. 10%

   (ii) Give an example to show that if f is not monotone, then f need not be continuous. 10%

3. Let $f : [1,\infty) \longrightarrow [0,\infty)$ be continuous and satisfy $\int_1^{\infty} (f(x))^2 \, dx < \infty$.

   (i) Show that $\int_1^{\infty} \frac{f(x)}{x} \, dx < \infty$. 10%

   (ii) Give an example to show that $\int_1^{\infty} \frac{f(x)}{\sqrt{x}} \, dx$ need not converge. 10%

4. Suppose $f : [0,1] \longrightarrow [0,1]$ is continuous and satisfies that $f(0) = 0$, $f(1) = 1$, and $f(f(x)) = x$ for all x in $[0,1]$.

   (i) Show that f is injective, and then conclude that f is strictly increasing by using some appropriate theorem. 10%

   (ii) Show that $f(x) = x$ for all x in $[0,1]$. 10%

5. Define $f : [0,\infty) \longrightarrow \mathbb{R} : f(x) = 2\int_0^x (\int_0^{x-u} e^{u^2+v^2} \, dv) \, du$.

   (i) Making a change of variables, show that $f(x) = \int_0^x F(t) \, dt$ for some appropriate function $F(t)$. 10%

   (ii) Find $f'(x)$. 10%