Please answer all questions (100 %)

1. (15 %) Let $f : \mathbf{R} \longrightarrow \mathbf{R}$ be a function defined by

$$f(x) = \int_{-2}^{x^3+x} \exp(xt^2)\,dt.$$

a) (5 %) Find the derivative of f at 0.
b) (5 %) Show that f is an one-to-one function on $(-1, +\infty)$.
c) (5 %) Find the derivative of f^{-1} at 2. (Hint: $f(0) = 2$.)

2. (20 %) Let $g : \mathbf{R} \longrightarrow \mathbf{R}$ be a function defined by

$$g(x) = \begin{cases} \dfrac{1-\cos x}{x^2}, & \text{if } x \neq 0; \\ \dfrac{1}{2}, & \text{if } x = 0. \end{cases}$$

a) (5 %) Show that g is a continuous function.
b) (5 %) Show that g is differentiable at 0.
c) (10 %) Prove that the integral $\displaystyle\int_0^{+\infty} g(x)\,dx$ converges.

3. (20 %)
a) (10 %) Using Taylor's Theorem to find a polynomial to approximate the function $h(x) = \cos^2 x$ defined on $(-1, 1)$ with error less than 0.15. Prove your result.
b) (10 %) Find all real numbers x for which the power series $\displaystyle\sum_{i=2}^{\infty} \frac{x^n}{n \log n}$ converges.

4. (25 %)
a) (15 %) Let $f : U \longrightarrow \mathbf{R}$ be a function defined by $f(x, y) = xy^2 + x^2y - xy + z^2$ where $U = \{(x, y, z) \in \mathbf{R}^3 \mid x^2 + y^2 + z^2 < 1\}$. Find local maxima and local minima of the function f on U if there exist any of them.
b) (10 %) Let $g : \mathbf{R} \longrightarrow \mathbf{R}$ be a function with $g(x, y) = xy^2$. Find global maxima of the function g on $F = \{(x, y) \in \mathbf{R}^2 \mid \dfrac{x^2}{4} + y^2 \leq 1\}$ if there exist any of them.

5) (20 %)
a) (10 %) Show that the equation $z^2x + y + \exp z = 0$ can be solved for z in terms of x and y in some neighborhood of $(0, 1, -1)$.
b. (10%) Let $S = \{(x, y, z) \in \mathbf{R}^3 \mid x^2 - y^2 - z = 0, x^2 + y^2 < 4\}$ be a surface. Compute the area of the surface S.

— **END** —

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