Linear Algebra

1993

Let V be a vector space over the field of real numbers R.

1. Let V_1 and V_2 be sunspaces of V. Show that
   (i) $V_1 \cap V_2$ is a subspace of V. (4%)
   (ii) $V_1 + V_2$ is a subspace of V. (4%)
   (iii) $\dim V_1 + \dim V_2 = \dim(V_1 + V_2)$, provided that V is finite dimensional and $V_1 \cap V_2 = \{0\}$. (8%)

2. Suppose that $\{u, v\} \subset V$. Show that $\{u, v\}$ is linearly independent if and only if $\{2u + v, u - 2v\}$ is linearly independent. (8%)

3. Let V be finite dimensional and $T\colon \mathrm{V} \to \mathrm{V}$ be linear. Show that T is *one-to-one* if and only if T is *onto*. You may apply the Dimension Theorem. (8%)

4. Let A be a real, symmetric matrix and $\lambda_1 \neq \lambda_2$ be two distinct eigenvalues of A. Let v_1 and v_2 be the eigenvectors corresponding to λ_1 and λ_2, respectively. Show that v_1 and v_2 are orthogonal. (8%)

5. Let
$$A = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix}.$$
Find an orthogonal matrix P and a diagonal matrix D such that $A = P^{\mathrm{T}} D P$ where P^{T} denote the transpose of P. (10%)

030

Probability

1. Let the pair of random variables (X, Y) has joint density function $f_{XY}(x, y) = 60x^2y$ on the triangle bounded by $x = 0$, $y = 0$ and $y = 1 - x$.
   Consider $Z = \begin{cases} \frac{X^2}{2Y} & \text{for } X \le \frac{1}{2} \\ 2Y & \text{for } X > \frac{1}{2} \end{cases}$. Determine
   (a) $E[Y|X = x]$ (b) $E[Z|X = x]$ (c) $E[Z]$. (18%)

2. Suppose that $(X_1, \ldots, X_r) \sim M(n; p_1, \cdots, p_r)$ is multinomial distributed.
   Determine
   (a) $\operatorname{var} X_r$ (b) $\operatorname{cov}(X_1, X_2)$. (10%)

3. Let $\{X_i : i \ge 1\}$ be *iid*, with $E[X_i] = 0$, $\operatorname{var}[X_i] = \sigma^2 > 0$. Let $Y_n = \sum_{i=1}^{n} X_i$.
   Prove that, for any $c > 0$, $\lim_{n \to \infty} P\{|Y_n| \ge c\} = 1$. (10%)

4. (a) Let $\{A_n\}$ be a sequence of events such that $\sum_{n=1}^{\infty} P(A_n) < \infty$.
   Let $A = \{\omega : \omega \in A_n \text{ for infinitely many } n\}$. Prove that $P(A) = 0$. (6%)

   (b) In coin-tossing, let $X_i = \begin{cases} 1 & \text{as the } i\text{th toss is head} \\ -1 & \text{as the } i\text{th toss is tail} \end{cases}$ be *iid* with
   $P\{X_1 = 1\} \ne \frac{1}{2}$. Let $Y_n = X_1 + \cdots + X_n$; $A_n = \{Y_n = 0\}$.
   $A = \{\omega : \omega \in A_n \text{ for infinitely many } n\}$. Show that $P(A) = 0$. (6%)