Linear Algebra

1993

Let V be a vector space over the field of real numbers R.

1. Let V_1 and V_2 be sunspaces of V. Show that
   (i) $V_1 \cap V_2$ is a subspace of V. (4%)
   (ii) $V_1 + V_2$ is a subspace of V. (4%)
   (iii) $\dim V_1 + \dim V_2 = \dim(V_1 + V_2)$, provided that V is finite dimensional and $V_1 \cap V_2 = \{0\}$. (8%)

2. Suppose that $\{u, v\} \subset V$. Show that $\{u, v\}$ is linearly independent if and only if $\{2u + v, u - 2v\}$ is linearly independent. (8%)

3. Let V be finite dimensional and $T\colon \mathrm{V} \to \mathrm{V}$ be linear. Show that T is *one-to-one* if and only if T is *onto*. You may apply the Dimension Theorem. (8%)

4. Let A be a real, symmetric matrix and $\lambda_1 \neq \lambda_2$ be two distinct eigenvalues of A. Let v_1 and v_2 be the eigenvectors corresponding to λ_1 and λ_2, respectively. Show that v_1 and v_2 are orthogonal. (8%)

5. Let
$$A = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix}.$$
Find an orthogonal matrix P and a diagonal matrix D such that $A = P^{\mathrm{T}} D P$ where P^{T} denote the transpose of P. (10%)

PART II.

1. Show that the following procedure is correct with the initial assertion

   AI : n > 0

   and the final assertion

   AF : $\forall j\ [1 \le j \le n \longrightarrow V[j] = 0]$:

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procedure ZERO;
begin
   i <— 1;
   while i ≤ n do
      begin
         V[i] <— 0;
         i <— i + 1
      end
end.
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10%

2. The Ackermann's function is defined on {0,1,2,3,...} X {0,1,2,3...} by

$$A(m, n) = \begin{cases} n + 1, & \text{if } m = 0, \\ A(m-1, 1), & \text{if } n = 0, \\ A(m-1, A(m, n-1)), & \text{otherwise.} \end{cases}$$

   Find the values of A(1, 3) and A(2, 5). 10%

3. Let $f : A \longrightarrow B$ be a function and define the relation R on the set A by

   $x \mathrel{R} y$ if and only if $f(a) = f(y)$,

   where x and y are in A. Show that R is an equivalence relation on A. 10%

4. Show that the Bubble sort for n elements is an $O(n^2)$ algorithm. 10%

5. Use Boolean algebra to design an adder(a device does the arithmetic addtion) and explain how it works. 10%