Ordinary Differential Equation

I. Consider the first order O.D.E.

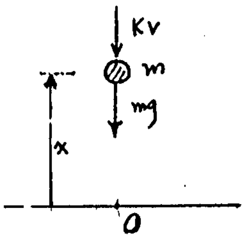
$$\frac{dy}{dx} = e^{-x} + y$$

(a) Sketch the direction field of the equation.

(b) Base on the direction field determine the behavior of y as $x \to +\infty$.

(c) Solve the equation, then compare with the results of (b).

II. A mass m is projected vertically upward from O with initial velocity v_0, the resistance of the air is proportional to the velocity as shown on the picture.



(a) Find the maximum height reached.

(b) What are the kinetic, potential and total energies?

(c) Is the system conservative? i.e., let $E(t)$ be the total energy, is $\frac{dE}{dt} = 0$?

(d) What is the role of the resistance coefficient K?

III. Determine the value of m and n so that $u(x,t) = t^m f(xt^n)$ is a solution of the Burger's equation

$$\partial_t u + u\partial_x u = \partial_{xx} u$$

and write down the equation satisfied by f. Hence obtain the solution which $f \to 0$ as $x \to \infty$ and $f(0) = -\frac{2}{\sqrt{\pi}}$. You will need the formula $\int_{-\infty}^{0} \exp -\frac{1}{4}t^2 dt = \sqrt{\pi}$.

IV. For a given function $y(t)$ defined by

$$y(t) = \int_0^t e^{-(t-\tau)} f(\tau) \sin(t-\tau) d\tau$$

(a) Show that y is the solution of the second order nonhomogeneous O.D.E.

$$y'' + 2y' + 2y = f(t), \qquad y(0) = y'(0) = 0$$

(b) In fact you can use the method of variation of parameter to derive the solution y.

(c) If we define

$$G(t,\tau) \equiv e^{-(t-\tau)} \sin(t-\tau)$$

show that G satisfies the differential equation

$$G'' + 2G' + 2G = \delta(t-\tau)$$

V. Let A be a constant $n \times n$ matrix then define the exponential $\exp A$ of the operator A by

$$\exp A \equiv I + A + \frac{A^2}{2} + \cdots$$

(a) Calculate the matrix $\exp A$ if the matrix is one of the following forms

$$1)\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \quad 2)\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad 3)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad 4)\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

(b) Rewrite the equation $x''(t) = x(t)$ as the first order system $X' = AX$, then draw the image of the circle $x^2 + (y-1)^2 = 1$ under the action of the transformation of the phase flow for the equation. Can you compute the area? Is the system conservative? Do you see the relation between area preserving, conservation of energy and the trace of the matrix A? 034