1. (a) Let $X \sim N(\theta, \sigma^2)$, and let g ge a differentiable function satisfying $E|g'(x)| < \infty$. Show that

$$E[g(x)(x-\theta)] = \sigma^2 E[g'(x)]. \qquad (10\%)$$

(b) Let $X_1, \ldots, X_n$ be iid $N(\theta, 1)$. Show that the uniformly minimum variance unbiased estimator of θ^2 is $\bar{X}^2 - \frac{1}{n}$. Calculate its variance, and show that it is greater than the Cramér–Rao Lower Bound. (10%)

2. (a) Suppose that $X_1, \ldots, X_n$ has a joint *pdf* $f(x_1, \ldots, x_n; \theta)$. State the definition of the pivotal quantity. (5%)

(b) Consider a random sample from a Parato distribution, $X_i \sim PAR(1, \theta)$, $i = 1, 2, \ldots, n$, i.e. the *pdf* of X_i is $f(x_i; \theta) = \frac{\theta}{(1+x)^{\theta+1}}$, for $x > 0$, $\theta > 0$. Use the pivotal quantity method to find a $100(1-\alpha)\%$ confidence interval of θ. (15%)

3. Let $X_1, \ldots, X_n$ be a random sample from the density

$$f(x|\theta) = \frac{1}{\theta} I_{(0,\theta)}(x).$$

Assume that a prior distribution of θ has a *pdf* $g(\theta) = I_{(0,1)}(\theta)$.

(a) For the loss function $\ell(t, \theta) = (t-\theta)^2$, find the Bayes estimator of θ. (10%)

(b) For the loss function $\ell(t, \theta) = \frac{(t-\theta)^2}{\theta^2}$, find the Bayes estimator of θ. (10%)

4. Let X_i be independently distributed as $N(i\Delta, 1)$, $i = 1, 2, \ldots, n$. Show that there exists a UMP test of $H_0 : \Delta \le 0$ against $H_a : \Delta > 0$, and determine it as explicity as possible. (20%)

5. Suppose that X is a continuous random variable with *pdf*

$$f(x; \theta) = \theta X^{\theta-1} \text{ if } 0 < x < 1, \text{ and zero otherwise.}$$

(a) Derive the generalized likelihood ratio test of $H_0 : \theta = \theta_0$ against $H_a : \theta \ne \theta_0$ based on a random sample of size n. (10%)

(b) Determine an approximate critical value for a size α test based on a large–sample approximation. (10%)

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