Probability

1. (10%) Find the expectation and variance of the random variable X if the distribution function of X is given by

$$F(x) = \begin{cases} 0, & \text{if } x < 0, \\ 1 - \frac{3}{5}e^{-x}, & \text{if } x \geq 0. \end{cases}$$

2. (15%) Let X be a random variable. Show that if $\mathrm{Var}(X) = 0$, then $P(X = EX) = 1$.

3. Let X be a random variable distributed as negative binomial with p.d.f. (or p.m.f.)

$$f(x; r, p) = p^r \binom{r+x-1}{x} (1-p)^x, \quad x = 0, 1, \ldots, \quad 0 < p < 1, r = 1, 2, \ldots,$$

and let $g(x)$ be a function with $-\infty < Eg(X) < +\infty$ and $g(-1) = 0$.

(a) (10%) Show that

$$E[(1-p)g(X)] = E\left[\frac{X}{r+X-1} g(X-1)\right].$$

(b) (10%) Use (a) to find the expectation of X.

4. Let $X_1, X_2, \ldots, X_n$ be independent random variables distributed as $P(\lambda_1)$, $P(\lambda_2)$, $\ldots$, $P(\lambda_n)$, respectively. Let $T = \sum_{j=1}^{n} X_j$ and $\lambda = \sum_{j=1}^{n} \lambda_j$.

(a) (10%) Show that T is distributed as $P(\lambda)$.

(b) (10%) Find the conditional distribution of X_j, given $T = t$. [Note that $P(\lambda)$ denotes the Poisson distribution with parameter λ.]

5. Let X and Y be two independent random variables distributed as $\mathrm{Beta}(\alpha, \beta)$ and $\mathrm{Beta}(\alpha+\beta, \gamma)$, respectively. Set $U = XY$ and $V = X$.

(a) (10%) Find the joint p.d.f. of U and V.

(b) (10%) What is the marginal distribution of U?

Note that the p.d.f. of $\mathrm{Beta}(\alpha, \beta)$ is given by

$$f(x; \alpha, \beta) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}, \quad 0 < x < 1, \quad \alpha, \beta > 0.$$

6. (15%) Let $\{X_n\}$ be a sequence of random variables with $P\{X_n = \pm\frac{1}{n}\} = \frac{1}{2}$. Show that $X_n \xrightarrow{a.s.} 0$.