Linear Algebra

1. Let $A = \begin{bmatrix} 1 & 3 & 3 & 2 \\ 2 & 6 & 9 & 5 \\ -1 & -3 & 3 & 0 \end{bmatrix}$ and $b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$
   (a) Find rank A. (5%)
   (b) Find a basis of $\{x \mid Ax = 0\}$, the null space of A. (6%)
   (c) Find a condition on b such that the linear system $Ax = b$ has solutions. (6%)

2. Let V be a vector space, $T : V \to V$ be linear, $R(T)$ be the range of T and $N(T)$ be the null space of T. Prove that
   (a) $T^2 = 0$ if and only if $R(T) \subset N(T)$; (7%)
   (b) if $T^2 = T$, then $V = R(T) \oplus N(T)$. (7%)

3. Let $A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$ and $u(t) = \begin{bmatrix} v(t) \\ w(t) \end{bmatrix}$.
   (a) Diagonalize A. (8%)
   (b) Solve $\begin{cases} \dfrac{du}{dt} = Au \\ u(0) = \begin{bmatrix} 2 \\ 4 \end{bmatrix} \end{cases}$ (8%)

4. Suppose A is a complex 10×10 matrix with characteristic polynomial $t(t-1)^9$, and I is the identity matrix.
   (a) Find $[A(A-I)]^{10}$. (5%)
   (b) Show that $(A-I)^{101} = -(A-I)^{100}$. (12%)

5. Let $C[0,1]$ be the vector space of continuous functions on $[0,1]$, and the inner product on $C[0,1]$ be $< f, g >= \int_0^1 f(t)g(t)\,dt$. Suppose $S = \{1, t\}$ and W is the subspace spanned by S.
   (a) Find an orthonormal basis for W by applying the Gram-Schmidt process to S. (8%)
   (b) Find the orthogonal projection of $h(t) = e^t$ on W. (8%)

6. Let A be a real symmetric $n \times n$ matrix, $A = [a_{ij}]$. We call A is positive definite if $x^T A x > 0$ for all nonzero column vector x in $\mathbf{R}^n$.
   (a) Show that if A is positive definite, then
   $$|a_{ij}| \le \frac{1}{2}(a_{ii} + a_{jj}) \quad \text{for all } i, j = 1, 2, \ldots, n.$$
   (8%)
   (b) Prove that A is positive definite if and only if there exists a matrix R, rank $R = n$, such that $A = R^T R$. (12%)