Answer all questions (100%)

1. (a) If B is bounded in $\mathbf{R}^m$ and $f: B \to \mathbf{R}^n$ is uniformly continuous, show that f is bounded on B. (10%)

   (b) Show that $f(x) = \tan x$ is not uniformly continuous on $[0, \frac{\pi}{2})$ (10%)

2. (a) Let $x_1 = 1$ and $x_{n+1} = (2 + x_n)^{\frac{1}{2}}$ for $n \in \mathbf{N}$. Show that $\lim_{n\to\infty} x_n$ exists. What is the limit? (10%)

   (b) Show that the convergence of $\sum_{n=1}^{\infty} a_n$ implies the convergence of $\sum_{n=1}^{\infty} \frac{\sqrt{a_n}}{n^p}$ if $a_n \geq 0$, and $p > \frac{1}{2}$ (10%)

3. (a) Let $a < c < b$ and $g(x) = \begin{cases} 0, & a \leq x \leq c \\ 1, & c < x \leq b \end{cases}$. Show that f is integrable with respect to g over $[a, b]$ if and only if $\lim_{x\to c^+} f(x) = f(c)$. (10%)

   (b) Find the Riemann–Stieltjes integral $\int_0^5 x^3 d(x^2 + [x])dx$. (10%)

4. (a) Show that $f(x) = \begin{cases} x \sin\frac{\pi}{x}, & 0 < x \leq 2 \\ 0, & x = 0 \end{cases}$ is continuous, but isn't a function of bounded variation on $[0, 2]$. (10%)

   (b) Compute the total variation of $f(x) = [x] - x$, $0 \leq x \leq 2$. (10%)

5. (a) Let $S = \{(x, t) : a \leq x \leq b,\ c \leq t \leq d\}$, and $f: S \to \mathbf{R}$ be a continuous function. Define $F: [c, d] \to \mathbf{R}$ by $F(t) = \int_a^b f(x, t)dx$. Show that F is continuous. (10%)

   (b) In (a), if f and its partial derivative $\frac{\partial f}{\partial t}$ are continuous on S then F has a derivative on $[c, d]$ and

$$F'(t) = \int_a^b \frac{\partial f(x, t)}{\partial t}\, dx.$$

(10%)