

Ordinary Differential Equation

- (1) Solve the following differential equations: (30%)

(a) $2y(t)y''(t) + y'^2(t) = 0.$

(b) $ty'(t) - y(t) = t^\gamma, \quad \gamma \geq 1.$

(c) $y''(t) - 4y(t) = f(t).$

(d) $dz = (x^2 + 2xy - y^2)dx + (x^2 - 2xy - y^2)dy.$

- (2) (Picard's iteration) Consider the differential equation (20%)

$$y'(t) = f(y(t), t), \quad f \in C(\mathbb{R} \times \mathbb{R}^+),$$

determined by a function y in a domain of the extended phase space \mathbb{R}^2 . Then the *Picard mapping* is defined by

$$(Ay)(t) = y_0 + \int_0^t f(y(\tau), \tau) d\tau, \quad y_0 = y(0),$$

and the *successive Picard approximations* is given by

$$\phi_0 = y_0, \phi_1 = A\phi_0, \dots, \phi_n = A\phi_{n-1} = A^n\phi_0, \quad \forall n = 1, 2, 3, \dots$$

- (a) Use the Picard's iteration to construct a sequence of approximate solutions of the following differential equation

$$y'(t) = k(y + 2), \quad y(0) = 1, \quad k \in \mathbb{R},$$

and prove that the limit of the approximate solution is the solution. (You need to give a rigorous proof!)

- (b) Discuss the long time behaviour, i.e., $t \rightarrow \infty$, of the solution.
(c) Discuss the behaviour of the solution as $k \rightarrow 0$.

- (3) Consider the linear system (20%)

$$\frac{dx}{dt} = \lambda x + y, \quad \frac{dy}{dt} = \mu y, \quad (x(0), y(0))^t = (x_0, y_0)^t$$

where λ and μ are real constants.

- (a) Find out and classify the critical point then determine whether it is stable or unstable.
 (b) Sketch the trajectories in the phase plane.
 (c) Solve the linear system.
- (4) Show that there is at most one solution of the initial value problem (10%)

$$y'' + e^y = 1, \quad \text{for } t > 0,$$

$$y(0) = 1, \quad y'(0) = 0.$$

- (5) Let S_t be the sphere $(\xi - x)^2 + (\eta - y)^2 + (\zeta - z)^2 = t^2$ and $f(\xi, \eta, \zeta)$ continuous. Prove rigorously that the function given by

$$u(x, y, z, t) = \frac{1}{4\pi} \iint_{S_t} \frac{f(\xi, \eta, \zeta)}{t} dS,$$

satisfies the initial value problem of the wave equation (20%)

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{\partial^2 u}{\partial t^2}$$

$$u(x, y, z, 0) = 0,$$

$$\frac{\partial}{\partial t} u(x, y, z, 0) = f(x, y, z).$$