PROBABILITY

1. Suppose (Ω, F, P) is a probability space and X is a random variable (r.v.) defined on Ω into R.
   (a) State the definition of the probability space (Ω, F, P). (5%)
   (b) State the definition of the r.v. X. (5%)
   (c) Suppose the probability density function (p.d.f.) of the r.v. X is $f_x(x)$. Let $Y=X^2$. Is Y a r.v.? If yes, find the p.d.f. of the r.v. Y, say f_y, in terms of $f_x(\cdot)$. (10%)

2. Let Y be a r.v. distributed as an uniform distribution, U(0, 1), and let F be a distribution function (d.f.).
   Define the r.v. X by $X=F^{-1}(Y)$, where $F^{-1}(y)=\inf\{x\in R;\ F(x)\geq y\}$.
   (a) Prove that: $F^{-1}(y)\leq t$ if and only if $y\leq F(t)$. (10%)
   (b) Prove that: The distribution function of X is F. (10%)

3. Suppose that X_1 and X_2 have a continuous joint distribution for which the joint p.d.f. is $f(x_1, x_2)=x_1+x_2$, for $0<x_1<1$, $0<x_2<1$; and zero otherwise.
   Define $U=X_1X_2$, $V=X_1/X_2$.
   (a) Prove or disprove that: $E(UV)=E(U)E(V)$. (10%)
   (b) Prove or disprove that: $E(U/V)=E(U)/E(V)$. (10%)

4. Suppose X is a Poisson r.v. with mean 20. Let $p=\Pr\{X\geq 26\}$.
   (a) Use the Markov inequality to obtain an upper bound on p.(5%)
   (b) Prove the one-sided Chebyshev inequality. That is,
   if X is a r.v. with mean 0 and finite variance σ^2, then for any a>0
   $\Pr\{X\geq a\}\leq\sigma^2/(\sigma^2+a^2)$.(5%)
   (c) Use the one-sided Chebyshev inequality to obtain an upper bound on p.(5%)
   (d) Use the central limit theorem to approximate p, answer it in terms of $\Phi(\cdot)$ which is the d.f. of the r.v. N(0, 1).(5%)

5. Suppose that $X_1, X_2, \ldots, X_n, \ldots$ is a sequence of random variables from a distribution with d.f. $F(x)=1-(1+x)^{-1}$, for x>0; and zero otherwise.
   Let $Y_n=nX_{(1)}$, where $X_{(1)}=\min\{X_1, X_2, \ldots, X_n\}$.
   (a) Find the d.f. of Y_n. (10%)
   (b) Find the limiting distribution of Y_n; i.e. find the d.f. of a r.v. Y for which Y_n converges in distribution to Y. (10%)