Answer all Questions

(1) (a) Given the area of the circular disk $\{(x,y): x^2+y^2 \le 1\}$ is equal to π, find the area of the elliptical disk given by $\{(x,y): 2x^2+2xy+5y^2 \le 1\}$. (10%)

(b) Let $B = \{(x,y): 0 \le x+y \le 2, 0 \le y-x \le 2\}$. Evaluate the integral

$$\iint_B (y^2-x^2)e^{\frac{x^2+y^2}{2}}\, d(x,y).$$

(10%)

2. (a) Show that $\int_0^{+\infty} e^{-x^2}\, dx = \frac{\sqrt{\pi}}{2}$. (10%)

(b) Show that $\Gamma(\frac{1}{2}) = \sqrt{\pi}$, where $\Gamma(\alpha) = \int_0^{+\infty} e^{-x}x^{\alpha-1}\, dx$ is the Gamma function. (10%)

3. (a) Show that $\sum_{n=1}^{\infty} \frac{\sin nx}{n}$ is convergent for all $x \in \mathbb{R}$. (10%)

(b) Show that $\sum_{n=1}^{\infty} \frac{\sin nx}{n}$ is uniformly convergent on $[a,b] \subseteq (0,2\pi)$. (10%)

4. (a) Let f be defined on $D \subseteq \mathbb{R}^p$ to $\mathbb{R}^q$, $p,q \ge 1$, and suppose f is uniformly continuous on D. If $\{x_n\}$ is a Cauchy sequence in D, show that $\{f(x_n)\}$ is a Cauchy sequence in $\mathbb{R}^q$. (10%)

(b) Suppose that $f:(0,1) \to \mathbb{R}$ is uniformly continuous on $(0,1)$. Show that f can be defined at $x=0$ and $x=1$ in such a way that it becomes continuous on $[0,1]$. (10%)

5. A set $\mathcal{F}$ of functions on $K \subseteq \mathbb{R}^p$ to $\mathbb{R}^q$ is said to be uniformly equicontinuous on K if, for each $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that if $x, y \in K$ and $\|x-y\| < \delta(\varepsilon)$, $f \in \mathcal{F}$, then $\|f(x)-f(y)\| < \varepsilon$. Now, let $\{f_n\}$ be a sequence of continuous functions on $\mathbb{R}$ to $\mathbb{R}^q$ which converges at each point of the set Q of rationals. If $\{f_n\}$ is uniformly equicontinuous on $\mathbb{R}$,

(a) show that $\{f_n\}$ converges at every point of $\mathbb{R}$. (10%)

(b) show that $\{f_n\}$ is uniformly convergent on every compact set of $\mathbb{R}$. (10%)