1. (a) The equation $p(x)y'' + Q(x)y' + R(x)y = 0$ is said be exact if there exists a function f such that $P(x)y'' + Q(x)y' + R(x)y = [P(x)y']' + [f(x)y]'$. Find a necessary and sufficient condition in terms of $P(x)$, $Q(x)$, $R(x)$ so that the above equation is exact. (7%)

   (b) Show that the equation $(1+x^2)y'' + xy' - y = 0$ is exact and find the general solution of the equation. (8%)

2. Determine the equilibrium points and classify each one as stable or unstable for the following equation

$$y' = y^3 - 3y^2 + 2y.$$

Justify your answer. (10%)

3. Let y_1 and y_2 be two solutions of the equation

$$y'' + p(x)y + q(x)y = 0 \qquad (x > 0)$$

with the Wronskian given by $W(y_1, y_2) = -\frac{2}{x}$. If $y_1 = x^{-1}$ is one of the solution, find p, q and y_2 and then solve the equation

$$y'' + p(x)y + q(x)y = \frac{\ln x}{x} \qquad (x > 0).$$

(15%)

4. Consider the solution of the equation

$$x^2y'' + xp(x)y' + \frac{5}{2}y = 0,$$

where $p(x) = \sum_{n=0}^{\infty} p_n x^n$ is analytic at 0. Find all the possible values of p_n, $n = 0, 1, 2, \ldots$, such that the solutions approach zero as $x \to 0$. (20%)

5. (a) Show that the eigenvalues of the boundary value problem

$$\begin{cases} [(1+\cos^2 x)y']' - e^x y + \lambda y = 0 \\ y(0) = y(\pi) = 0 \end{cases}$$

   are positive. (10%)

   (b) Find the Green's funciton for the boundary value problem

$$\begin{cases} y'' + y = -f \\ y(0) = 0, \quad y(1) = 0. \end{cases}$$

(15%)

6. (a) Let $\Phi(t)$ denote the fundamental matrix of the equation $x'(t) = Ax(t) \quad (t \geq 0)$ satisfying the condition that $\Phi(0) = I$, where A is a $n \times n$ matrix, I the identity matrix and $x(t) \in \mathbb{R}^n$. Show that $\{\Phi(t),\ t \geq 0\}$ is a semigroup. (7%)

   (b) Suppose that $A = (a_{ij})$ with $a_{ij} = \frac{1}{i(j+1)} \quad (i, j = 1, 2, \ldots, n)$. Find $\det(\Phi(1))$. (8%)