1. If $x \sin \pi x = \int_0^{x^2} f(t)\,dt$ where f is a continuous function, find $f(4)$. (10%)

2. Show that of all the isosceles triangles with a given perimeter, the one with the greatest area is equilateral. (12%)

3. A number x_0 is called a fixed point of a function f if $f(x_0) = x_0$.

   (a) Show that if $f'(x) < 1$ for all $x \in \mathbb{R}$, then f has at most one fixed point. (10%)

   (b) Construct a function g such that $g'(x) < 1$ for all $x \in \mathbb{R}$ and g has no fixed point. **Hint**: use $\tan^{-1} x$. (10%)

4. If $f(t)$ is continuous for $t \geq 0$, the Laplace transform of f is the function F defined by $F(s) = \int_0^{\infty} f(t)e^{-st}\,dt$. Now suppose that $0 \leq f(t) \leq Me^{at}$ and $0 \leq f'(t) \leq Ke^{at}$ for $t \geq 0$, where f' is continuous. If the Laplace transform of $f(t)$ is $F(s)$, and the Laplace transform of $f'(t)$ is $G(s)$. Show that $G(s) = sF(s) - f(0)$ for $s > a$. (12%)

5. (a) Let $a_1 = \sqrt{2}$ and $a_{n+1} = \sqrt{2 + \sqrt{a_n}}$ for $n = 1, 2, 3, \ldots$, show that $\{a_n\}$ is convergent. (8%)

   (b) Prove that if $a_n \geq 0$ for all n and $\sum_{n=1}^{\infty} a_n$ is convergent, then $\sum_{n=1}^{\infty} \frac{\sqrt{a_n}}{n}$ is convergent. (8%)

6. Let $T(x,y) = x^2 + xy + y^2 + x$ be the temperature function of the region $\{(x,y) : x^2 + y^2 \leq 1,\ y \geq 0\}$. Find the maximal and the minimal temperature of this region. (15%)

7. Let $R = \{(x,y) : x^2+y^2 \leq 2,\ 0 \leq x \leq 1,\ y \geq 0\}$ and $f(x,y) = \begin{cases} e^{x^2+y^2} & \text{if } x \leq y \\ 2e^{(1-y)^2} & \text{if } x > y. \end{cases}$

   Evaluate the integral $\iint_R f(x,y)\,dA$. (15%)