PROBABILITY

1. Let $(X_n)_{n\geq 1}$be a sequence of independent random variables with the distribution $P(X_n = 1) = p, P(X_n = 0) = 1 - p$. Define $T : \Omega \to N \cup \{+\infty\}$ by $T(\omega) = \inf\{n \mid X_n(\omega) = 1\}$, if $\{n \mid X_n(\omega) = 1\} \neq \phi$ and $T(\omega) = +\infty$, otherwise. Find $P(T = +\infty)$. (16%)

2. Let $(X_n)_{n\geq 1}$be a sequence of i.i.d. random variables uniformly distributed on $[0, 1]$. Define $Z_n = \min(X_1, X_2, \cdots, X_n)$. Does the sequence $(nZ_n)_{n\geq 1}$ converge in distribution ? Why ? (16%)

3. Let $(X_n)_{n\geq 1}$be a sequence of random variables such that $P\left(X_n = \frac{k}{n}\right) = \frac{1}{n}, 1 \leq k \leq n$. Does it converge in distribution ? Why ? (16%)

4. Let $X_1, X_2, \cdots, X_m$ be m independent random variables with values in $N \cup \{0\}$ and with a common distribution $P(X_i = k) = p_k\ (k \geq 0)$. Define

$$\gamma_n = \sum_{k=n}^{\infty} p_k$$

Show that $E[\min(X_1, X_2, \cdots, X_m)] = \sum_{n=1}^{\infty} \gamma_n^m$. (16%)

5. Use the methods in the probability theory to find

$$\lim_{n\to+\infty} e^{-n} \sum_{k=0}^{n} \frac{n^k}{k!}$$ (16%)

6. In probability space $(\Omega, \mathcal{B}, P)$, let $\Omega = [0,1]$, $\mathcal{B}$ = Borel field, P = Lebesgue measure, $X(\omega) = \sin(2\pi\omega)$ and $Y(\omega) = \cos(2\pi\omega)$ where $\omega \in [0, 1]$.

   (a) Are X and Y uncorrelated ? (10%)

   (b) Are X and Y independent ? (10%)