I. Notations and definitions

In the following problem set, the symbols $\mathbb{N}$ and $\mathbb{R}$ will be reserved for the set of all positive integers and the set of all real numbers, respectively.

We shall fix a field K and denote by V, W and W' finite dimensional vector spaces over K and we shall use $\dim V$ for the dimension of V. For any linear transformation $f : V \to W$, $\ker f$ is the kernel of f and $\operatorname{Im} f$ is the image of f.

The letters n and r will denote natural numbers, and $\mathrm{Mat}_n(K)$ ($\mathrm{Mat}_r(K)$) will denote the set of all n by n (r by r, resp.) matrices over K. We call two matrices $A, B \in \mathrm{Mat}_n(K)$ *similar* if there exists an invertible matrix $P \in \mathrm{Mat}_n(K)$ such that $P^{-1}AP = B$. A matrix $A \in \mathrm{Mat}_n(K)$ is called *nilpotent* if $A^s = 0$ for some $s \in \mathbb{N}$.

II. Problems

(1) Let E be a subset of V and $\langle E \rangle = \{\sum_{\alpha \in K; v \in V} \alpha v \mid \text{only finitely many } \alpha \text{ are nonzero}\}$. Show that $\langle E \rangle$ is a subspace of V and that $\langle E \rangle = \bigcap\{W \mid W \text{ is a subspace and } E \subseteq W\}$. 10%

(2) Let $f : V \to W$ be a linear transformation. Then there exist bases $\{u_1, \ldots, u_m\}$ of V and $\{v_1, \ldots, v_n\}$ of W and a positive integer r, $r \le m$ and $r \le n$, such that $f(u_i) = v_i$ for $i = 1, \ldots, r$ and $f(u_i) = 0$ for $i = r+1, \ldots, m$. 10%

(3) Let $f : V \to W$ and $g : V \to W'$ be linear transformations such that $\ker g \subseteq \ker f$. Show that there exists a linear function $h : W \to W'$ such that $h \circ g = f$. (Hint. Consider extending a basis of $\ker g$ to a basis of V and remember that $\dim V = \dim(\ker g) + \dim(\operatorname{Im} g)$.) 15%

(4) (i) Let $\{x_1, \ldots, x_m\}$ be a basis of V. Suppose that $\alpha_1, \ldots, \alpha_m \in K$ are pairwise distinct scalars. If $f : V \to V$ is a linear transformation such that $f(x_i) = \alpha_i x_i$ for $i = 1, 2, \ldots, m$, and if $g : V \to V$ is a linear transformation with the property that $f \circ g = g \circ f$, then there exist scalars $\beta_1, \ldots, \beta_m \in K$ such that $g(x_i) = \beta_i x_i$ for $i = 1, 2, \ldots, m$. 10%

(ii) Show that if $A \in \mathrm{Mat}_m(K)$ and $AB = BA$ for all $B \in \mathrm{Mat}_m(K)$, then A is a scalar diagonal matrix. 10%

(iii) Show that if $A, B, C \in \mathrm{Mat}_2(K)$, then $(AB - BA)^2$ commutes with C. 5%

(5) If $A, B \in \mathrm{Mat}_n(K)$ with A invertible, then the matrix $A + rB$ is invertible for all but finite number of $r \in K$. 15%

(6) Let $A \in \mathrm{Mat}_n(K)$ be nilpotent. Then A is similar to a matrix of the form 15%

$$\begin{pmatrix} W & 0 \\ 0 & 0 \end{pmatrix},$$

where $W \in \mathrm{Mat}_r(K)$, $1 \le r \le n$, is of the form

$$W = \begin{pmatrix} 0 & 1 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 1 & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \ddots & 1 \\ 0 & 0 & 0 & 0 & \ldots & 0 \end{pmatrix}.$$

(7) For each of the following matrices A and B, determine whether or not it is diagonalizable over $\mathbb{R}$. If it is diagonalizable, then find the change-of-coordinate matrix which make it a diagonal matrix, otherwise, explain why it cannot be diagonalized. 10%

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{pmatrix}, \qquad B = \begin{pmatrix} 3 & 1 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{pmatrix}.$$