(1) (15%)

(1a) Please define what is a *compact set*?

(1b) Use (1a) to verify whether the following sets are compact or not?

$$(0,1), \quad [0,1), \quad (0,1], \quad [0,1].$$

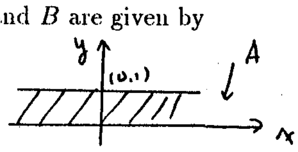
(1c) Please describe the open, connected and compact sets in $\mathbf{R}$.

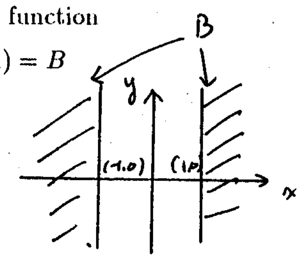
(2) (10%) Prove or disprove there exists a continuous function

$$f : \mathbf{R}^2 \mapsto \mathbf{R}^2 \qquad \text{such that} \quad f(A) = B$$

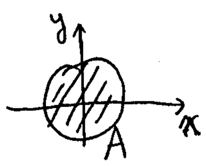
where A and B are given by

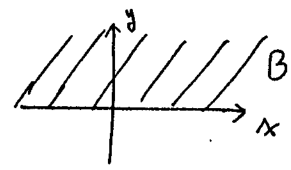
(2a)





(2b)





(3) (10%) Given a set

$$A = \left\{ x^{\alpha} \sin \frac{1}{x} \,\middle|\, x \in (0, 2\pi] \right\} \qquad \alpha \geq 0$$

Is A a connected set ? What is the colsure of A, i.e., $\bar{A}$?

(4) (10%)

(4a) Given a function $\{x^{\alpha} | x \in [0,1]\}, \alpha > 0$. Prove or disprove it is uniform continuous?

(4b) Given a function $\{x^{\alpha} | x \in [0,\infty)\}, \alpha > 0$. Prove or disprove it is uniform continuous?

(5) (15%) Given a sequence of trigonometric functions

$$\left\{ \sin^2 nx \,\middle|\, x \in [0, 2\pi] \right\} \qquad n = 1, 2, 3 \cdots$$

Does it converge uniformly? What is the limit of the integral

$$\int_0^{2\pi} \sin^2 nx f(x) dx \qquad f \in C[0, 2\pi]$$

as $n \to \infty$.

(6) (20%) Given an improper integral

$$\int_0^{\infty} \frac{\sin x}{x} dx$$

Does it converge? Is it absolutely convergent? If converge please compute the integral.

(7) (20%) Prove that the double improper integral

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(ax^2 + 2bxy + cy^2)} dx dy \qquad a > 0, \quad ac > b^2$$

converges. Could You evaluate the integral? (You must explain every step rigourously.)