(1) Let f be a differentiable function such that $f(x) > 0$ for all $x \in \mathbb{R}$. Suppose that $f(x+y) = f(x)f(y)$ for all $x, y \in \mathbb{R}$. Let $g = f^{-1}$ be its inverse function. Prove that

(a) $g(xy) = g(x) + g(y)$ for all $x, y \in \mathbb{R}$; 7%

(b) $g(\frac{1}{x}) = -g(x)$ for all $x > 0$; 7%

(c) $g'(x) = \dfrac{g'(1)}{x}$ for all $x > 0$. 7%

(2) Let $f(x) = x^3 + ax^2 + bx + c$. Prove that

(a) $f(x)$ has a point of inflection; 7%

(b) $f(x)$ has a local maximum and local minimum if and only if $a^2 > 3b$; 7%

(c) If $f(x)$ has a local maximum x_M and local minimum x_m, then its inflection point is at $\dfrac{x_m + x_M}{2}$. 7%

(3) Let $\sum_{n=0}^{\infty} \dfrac{k^n n! n!}{(2n)!}$ be infinite series for $k = 1, 2, 3, 4, 5, \cdots$. For what k does the series converge? For what k does the series diverge? Explain why. 15%

(4) Let $f(x)$ be a smooth function defined on the interval $[a, b]$ with $f(a) = f(b) = 0$. Prove that 15%

$$\int_a^b \frac{f(x)^2}{4r(x)^2}\,dx \le \int_a^b f'(x)^2\,dx,$$

where $r(x) = \min\{|x-a|, |x-b|\}$. (Hint. You may find the equation $f'(x) = \sqrt{x}\left(\dfrac{f(x)}{\sqrt{x}}\right)' + \dfrac{1}{2x}f(x)$ useful.)

(5) Let f, g be two positive functions defined on the interval $[a, \infty)$ satisfying that $f(a) > g(a)$, $f'(a) > g'(a)$, and $\dfrac{f''}{f}(x) > \dfrac{g''}{g}(x)$ for all $x \ge a$. Prove that $f(x) > g(x)$ for all $x \ge a$. 14%

(6) If the temperature of a plate at the point (x, y) is $T(x, y) = 1 + x^2 - y^2$, find the path of a heat-seeking particle (which always moves in the direction of greatest increase in temperature) would follow if it starts at $(1, 0)$. 14%