Notations:

R = the set of real numbers.

$P_n(R)$ = the set of all polynomails with coefficients from R having degree less than or equal to n.

1. Let u and v be linearly independent vectors of a vector space over a field F. If a, b, c, and d are elements of F, prove that the vectors $au + bv$ and $cu + dv$ are linearly independent iff $ad - bc \neq 0$. 10%

2. Let V be the vector space of all functions from R into R; let W_1 be the subspace of even functions, $f(-x) = f(x)$; let W_2 be the subspace of odd functions, $f(-x) = -f(x)$. Prove that $V = W_1 \oplus W_2$. 10%

3. Let $T : P_2(R) \longrightarrow P_3(R)$; $T(f(x)) = xf(x) + f'(x)$.
   ($f'(x)$ is the formal derivative of $f(x)$)
   a) Prove that T is a linear transformation. 5%
   b) Find bases for $N(T)$ and $R(T)$. 10%
      ($N(T)$ is the null space of T and $R(T)$ is the range of T)
   c) Determine whether T is one-to-one or onto. 5%

4. Suppose that $T : R^2 \longrightarrow R^3$ is linear and that 10%
   $T(1, 1) = (1, 0, 2)$ and $T(2, 3) = (1, -1, 4)$.
   What is $T(8, 11)$? Is T one-to-one ? Justify your answer.

5. Let $T: P_2(R) \longrightarrow P_2(R)$ define by $T(f) = f'' + 2f' - f$. 15%
   (f'' is the second formal derivative of f)
   Prove that T is invertible and compute T .

6. Let $T: P_2(R) \longrightarrow P_2(R)$ defined by 10%
   $$T(f) = f(0) + f(1)(x + x^2).$$
   Find a basis β such that $[T]_\beta$ is a diagonal matrix.
   ($[T]_\beta$ is the matrix that represents T in the ordered basis β)

7. Disprove the following statements.
   a) Similar matrices always have the same eigenvalues. 5%
   b) Similar matrices always have the same eigenvectors. 5%

8. Let 15%
   $$A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}.$$
   Find an orthogonal matrix P and a diagonal matrix D such that $P^* AP = D$. (P^* is the conjugate transpose of P)