Answer all questions (you must justify your answers)

1. Let $E \subseteq \mathbb{R}^3$ and $\overline{E}$ be the closure of E, prove or disprove that
   (a) if E is connected, then $\overline{E}$ is connected; (8%)
   (b) if $\overline{E}$ is connected, then E is connected; (8%)
   (c) if E is compact and $f : \mathbb{R}^3 \to \mathbb{R}^3$ is continuous, then f is uniformly continuous on E. (10%)

2. Suppose $f(x,y) = \begin{cases} \dfrac{x^2 y}{x^4 + y^2}, & \text{if } (x,y) \neq (0,0); \\ 0, & \text{if } (x,y) = (0,0). \end{cases}$
   (a) Find $f_x(0,0)$ and $f_y(0,0)$ if they exist; (6%)
   (b) Is f differentiable at $(0,0)$? (8%)
   (c) Does $\int_0^1 dx \int_0^1 f(x,y)\,dy = \int_0^1 dy \int_0^1 f(x,y)\,dx$? (10%)

3. Let $f_n : [0,1] \to \mathbb{R}$ be defined by $f_n(x) = \dfrac{x}{(1+x)^n}$, for $n = 0, 1, 2, \ldots$.
   (a) Prove that $\sum_{n=0}^{\infty} f_n(x)$ is convergent for all $x \in [0,1]$; (6%)
   (b) Is it uniformly convergent on $[0,1]$? (6%)
   (c) Does $\int_0^1 \sum_{n=0}^{\infty} f_n(x)\,dx = \sum_{n=0}^{\infty} \int_0^1 f_n(x)\,dx$? (8%)

4. Evaluate the integral $\iint_D \sin\left(\frac{x-y}{x+y}\right) dA$
   where $D = \{(x,y) \,|\, 0 < y < x < 1 - y\}$. (15%)

5. Suppose f and g are continuous functions on $[a,b]$.
   (a) Show that if $g(x) \geq 0$ for all $x \in [a,b]$, then there exists $c \in [a,b]$ such that $\int_a^b f(x)g(x)\,dx = f(c) \int_a^b g(x)\,dx$. (8%)
   (b) Given an example of f and g to show the result in (a) is fales when the condition "$g(x) \geq 0$" is dropped. (7%)