1. Assume the temperature $u(x,y)=\ln(x^2+y^2)$ is a function of position (x,y) for the region $\{(x,y)|x^2+y^2\geq 1\}$. A particle moves in the region. The position of the particle at time t is $(x^*(t),y^*(t))$, where $x^*(t)=t\cos(t)$ and $y^*(t)=t\sin(t)$.
   (a) Find the unit tangent vector $T(t)$ and the unit normal vector $N(t)$ for the path of the particle. 4%
   (b) Find the velocity and the acceleration of the particle. 4%
   (c) Find the temperature change rate at the particle, i.e. $\frac{d}{dt}u(x^*,y^*)$. 4%
   (d) Find the gradient of the temperature at (x,y), i.e. $\nabla u(x,y)$. 4%
   (e) Show that the divergence of ∇u is zero, i.e. $\nabla\cdot\nabla u=0$. 4%
   (f) The heat flux is $D\nabla u(x,y)$ where is D a positive constant. Find the total heat flow which diffuse from outside into the region. 5%

2. Let $\sum_{i=1}^{\infty}a_i$ be a real series.
   (a) State the definition of "$\sum_{i=1}^{\infty}a_i=L$" (the series converges to L). 5%
   (b) Prove that "If $\sum_{i=1}^{\infty}a_i$ converges than $\lim_{i\to\infty}a_i=0$. 5%
   (c) Prove that "If $\sum_{i=1}^{\infty}|a_i|$ converges than $\sum_{i=1}^{\infty}a_i$ converges. 5%

3. Let $f(x)=\int_1^{x}\exp(t^3)dt$
   (a) Find $f(1)$. 5%
   (b) Find $f'(1)$. 5%
   (c) Find $\int_0^1 xf(x)dx$. 5%

4. Let $f(x)$ be twice differentiable on an open interval containing a and b, and $|\frac{d^2}{dx^2}f(x)|<M$ for $x\in[a,b]$. Let $g(x)$ be the linear interpolation for $f(x)$; i.e. $g(x)=f(a)+\frac{f(b)-f(a)}{b-a}(x-a)$. Prove that
   $|f(x)-g(x)|<\frac{|b-a|}{4}M$ for $x\in[a,b]$. 15%

(背面仍有題目,請繼續作答)

5. Evaluate the following integrals

(a) $\int_0^{\pi} \sin^3 x dx$. 5%

(b) $\int_{-\infty}^{0} e^{2x} \cos x dx$. (improper integral) 5%

(c) $\int_{-1}^{1} \int_0^{\sqrt{1-x^2}} \frac{y}{\sqrt{x^2+y^2}} dy dx$. (double integral) 5%

6. Evaluate the following limits

(a) $\lim_{x\to\infty} x \sin\frac{1}{x}$. 5%

(b) $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{(1+\frac{i}{n})n}$. 5%

(c) $\lim_{x\to 0} \frac{\sin^{-1}(a+x) - \sin^{-1}(a-x)}{x}$. 5%