1. Find the density function of $Y = \sin^{-1} X$ when

   (a) X is uniformly distributed on $[0, 1]$. (10%)

   (b) X is uniformly distributed on $[-1, 1]$. (10%)

2. Let $X_1, X_2, \ldots$ be independent random variables which are uniformly distributed on $[0, 1]$. Let $M_n = \max\{X_1, X_2, \ldots, X_n\}$. Show that $n(1 - M_n)$ converges in distribution to X where X is exponentially distributed with parameter 1. (20%)

3. (a) Show that $\int_{-\infty}^{\infty} \exp(-x^2)\, dx = \sqrt{\pi}$, and deduce that

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\{-\frac{(x-\mu)^2}{2\sigma^2}\}, \quad -\infty < x < +\infty$$

   is a density function if $\sigma > 0$. (5%)

   (b) Calculate the mean and variance of a standard normal random variable. (5%)

   (c) Show that the $N(0, 1)$ distribution function Φ satisfies

$$(x^{-1} - x^{-3})\exp(-\frac{1}{2}x^2) < \sqrt{2\pi}\{1 - \Phi(x)\} < x^{-1}\exp(-\frac{1}{2}x^2),$$

   for $x > 0$. (10%)

4. Let $Y_1, Y_2, \ldots$ be independent identically distributed random variables, each of which can take any value in $\{0, 1, \ldots, 9\}$ with equal probability $\frac{1}{10}$. Let

$$X_n = \sum_{i=1}^{n} Y_i 10^{-i}.$$

   Show by the use of characteristic functions that X_n converges in distribution to the uniform distribution on $[0, 1]$. (20%)

5. Let $(X_n)_{n \geq 1}$ be a sequence of independent identically distributed random variables with mean zero and $E(X_1^4) < +\infty$ Prove that $n^{-1}\sum_{i=1}^{n} X_i \xrightarrow{a.s.} 0$. (20%)