Answer all questions and justify your answers

1. Determine the values of p and q for which the following integrals converge: (24%)

   (a) $\int_0^{\frac{\pi}{2}} x^p(\sin x)^q\,dx$ (b) $\int_1^2 (\ln x)^p\,dx$

   (c) $\int_0^1 x^p(-\ln x)^q\,dx$.

2. (a) Let $\lim_{n\to\infty} x_n$ exist and let $y_n = \dfrac{x_1+\cdots+x_n}{n}$, $n\in\mathbb{N}$.
   Prove that $\lim_{n\to\infty} y_n$ exists. (10%)

   (b) Let $\sum_{n=1}^{\infty} a_n$ be convergent and let $c_n = \dfrac{a_1+2a_2+\cdots+na_n}{n(n+1)}$.
   Prove that $\sum_{n=1}^{\infty} c_n$ converges and equals $\sum_{n=1}^{\infty} a_n$. (10%)

3. Evaluate the following integrals: (16%)

   (a) $\int_0^{\infty} (\frac{\sin x}{x})^2\,dx$ (Hint: $\int_0^{\infty} \frac{\sin tx}{x}\,dx = \frac{\pi}{2}, t>0$)

   (b) $\int_0^{\infty} \frac{\cos y}{1+y^2}\,dy$ (Hint: Set $F(t) = \int_0^{\infty} \frac{\sin tx}{x(x^2+1)}\,dx$)

4. Let $f:[0,1]\to\mathbb{R}$. The nth Bernstein polynomial for f is defined to be
   $B_n(x) = \sum_{k=0}^{n} f(\frac{k}{n})\binom{n}{k}x^k(1-x)^{n-k}$

   (a) Prove that B_n converges uniformly on $[0,1]$ to f when f is continuous on $[0,1]$. (10%)

   (b) Prove that if $g:[a,b]\to\mathbb{R}$ is continuous, then g can be uniformly approximated by polynomials. (10%)

5. Every rational x can be written in the form $x=\frac{m}{n}$, where $n>0$ and m,n are integers without any common divisors, when $x=0$, take $n=1$.
   Consider function $f:\mathbb{R}\to\mathbb{R}$ by $f(x) = \begin{cases} 0, & x \text{ irrational} \\ \frac{1}{n} & x=\frac{m}{n} \end{cases}$

   (a) Prove that f is continuous at every irrational point. (10%)

   (b) Prove that f is discontinuous at every rational point. (10%)