Please work out all six problems.

1.(20%) Let y_1 and y_2 be two solutions of the second order differential equation $y''+p(x)y'+q(x)y=0$ on an open interval I, where $p(x)$ and $q(x)$ are continuous in I. Show that
(a) if y_1 and y_2 are linear dependent, then the Wronskian of y_1 and y_2 $W(y_1,y_2)\equiv 0$ on I;
(b) if y_1 and y_2 are linear independent, then the Wronskian $W(y_1,y_2)\neq 0$ at each point of I.

2.(20%) Show that the initial value problem

$$\begin{cases} y''(x) = -y^3(x) \\ y(0) = 1 \\ y'(0) = 0 \end{cases}$$

admits a unique solution defined for every $x \in \mathbf{R}$. Also, show that such a solution is a periodic function.

3.(20%) Let $a(t)$ be a real-valued continuous function on $[0,\infty)$ with

$$\int_0^\infty |a(t)|dt < \infty.$$

Prove that the solution to the following initial value problem

$$\begin{cases} y''(t) + y(t) = a(t)y(t) \\ y(0) = 1, \quad y'(0) = 0 \end{cases}$$

is bounded on $[0,\infty)$.

4.(20%) Solve the differential equation

$$y = xy' + (y')^2 \quad \text{or} \quad y = xp + p^2$$

where $p = y'$ by differentiating the equation with respect to x. You need to find the general solution and all singular solutions.

5.(10%) Let $x(t), y(t)$ be the solution of

$$\begin{cases} x' = y + x^2 \\ y' = x + y^2 \end{cases}$$

with the given initial condition $(x(t_0), y(t_0))$. Assume that $x(t_0) \neq y(t_0)$. Then show that $x(t)$ is never equal to $y(t)$ for all t.

6.(10%) Show that all solutions $x(t), y(t)$ of

$$\begin{cases} x' = y(e^x - 1) \\ y' = x + e^y \end{cases}$$

which start in the right half plane ($x > 0$) must remain there for all time.

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