Notations and Definitions:

$\mathbb{R}^n$: set of n-dimensional real vectors.
$\mathbb{R}^{n\times n}$: set of $n \times n$ real matrices.
$\wp_n(\mathbb{R})$: set of real polynomials of degree $\leq n$.
A^T: the transpose of the matrix A.
$A \in \mathbb{R}^{n\times n}$ is positive definite if $x^T Ax > 0$ for any nonzero $x \in \mathbb{R}^n$.

Problems:

1. Let $\mathcal{V}$ be an m ($m \leq n$) dimensional subspace of $\mathbb{R}^n$, $P \in \mathbb{R}^{n\times n}$ be a projection on $\mathcal{V}$, that is, $Px \in \mathcal{V}$ for any $x \in \mathbb{R}^n$ and $Pv = v$ for any $v \in \mathcal{V}$.
   (i) Show that $\det P = 0$. 5%
   (ii) Let $V_m = \{v_1, \ldots, v_m\}$ form an orthonormal basis of $\mathcal{V}$. Find a project P on $\mathcal{V}$ and represent P in a matrix form. 10%
2. Let $x_0 < x_1 < \cdots < x_n$ be $n+1$ distinct real numbers and $y_k \in \mathbb{R}$, $k = 0, 1, \ldots, n$. Show that there is a unique polynomial $p(x) \in \wp_n(\mathbb{R})$ such that $p(x_k) = y_k$, $k = 0, 1, \ldots, n$. 10%
3. Let $A = A^T$, B, $D = D^T \in \mathbb{R}^{n\times n}$ and $I \in \mathbb{R}^{n\times n}$ be the identity matrix.
   (i) Assume that A is positive definite. Show that if $D - B^T A^{-1} B$ is positive definite, then $M = \begin{bmatrix} A & B \\ B^T & D \end{bmatrix}$ is also positive definite. 10%
   (ii) Verify that if $\gamma > \|B\|_2^2$, then $M = \begin{bmatrix} I & B \\ B^T & \gamma I \end{bmatrix}$ is also positive definite. Here $\|B\|_2^2 = \sup_{\substack{x \in \mathbb{R}^n \\ x^T x = 1}} (x^T B^T Bx)$. 10%
4. Assume that $A \in \mathbb{R}^{n\times n}$ is fixed. Let T be a linear operator on $\mathbb{R}^{n\times n}$ defined by $T(B) = AB$. Show that the minimal polynomial for T is the minimal polynomial for A. 10%
5. Let $\mathcal{U}$ be an inner product space consisting of continuous complex-valued functions on the interval $0 \leq x \leq 1$ with the inner product

$$(f|g) = \int_0^1 f(x)\overline{g(x)}dx \quad \text{for any } f, g \in \mathcal{U}.$$

   (i) Show that $h_k(x) = e^{2\pi ikx}$, $k = \pm 1, \pm 2, \ldots$ are mutually orthogonal. Here $i = \sqrt{-1}$. 5%
   (ii) Verify the Bessel's inequality

$$\sum_{k=-n}^{n} \left| \int_0^1 f(t)e^{2\pi ikt}dt \right|^2 \leq \int_0^1 |f(t)|^2\, dt \quad \text{for } f \in \mathcal{U}.$$

10%

6. Let

$$\mathcal{W} = \left\{f : [0,1] \to \mathbb{R} \mid f \in C^2([0,1]) \text{ and } f(0) = 0 = f(1)\right\}$$

be an inner product space with the inner product

$$(f|g) = \int_0^1 f(x)g(x)dx \quad \text{for any } f, g \in \mathcal{W}.$$

Here $f \in C^2([0,1])$ means that f is defined on $[0,1]$ and its second derivative is also defined and continuous on $[0,1]$. Let D^2 be an operator on $\mathcal{W}$ defined by

$$D^2(f) = \frac{d^2 f}{dx^2} \quad \text{for } f \in \mathcal{W}.$$

(i) Show that D^2 is self-adjoint. 10% (Hint: Use integration by parts!)

(ii) Show that D^2 is positive definite, i.e., $(D^2 f|f) > 0$ for any nonzero function $f \in \mathcal{W}$. 10%

7. Let $T : \wp_2(\mathbb{R}) \longrightarrow \wp_2(\mathbb{R})$ be define by $T(f) = f(0) + f(1)(x + x^2)$. Show that T is diagonalizable. 10%

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