Entrance Exam, Advanced Calculus, April, 1999

Show all works

1. **(a)**[10] State the fundamental theorem of calculus.

(b)[10] Evaluate the integral $\int_0^1 \left(\int_x^1 \frac{\sin y}{y}\,dy\right)dx$.

2. **(a)**[10] Discuss the convergence of the integrals

$$\int_0^1 \frac{1}{x^p}\,dx \quad \text{and} \quad \int_1^\infty \frac{1}{x^p}\,dx.$$

(b)[10] Soppose that the series $\Sigma_{n=1}^{\infty} a_n$ converges and for each $a_n \geq 0$. Discuss the convergence of the series

$$\Sigma_{n=1}^{\infty} \sqrt{a_n}\, n^{-p}, \quad p \in \mathbf{R},$$

on which interval the series converges and on which interval the series may or may not diverge. If it is in the latter case, please give examples.

3. **(a)** Suppose that $\{f_n(x)\}$ and $\{g_n(x)\}$ converge uniformly on the interval $[a,b]$.

(1)[5] Show that the sequence $\{f_n(x)+g_n(x)\}$ converges uniformly on the interval $[a,b]$.

(2)[5] Does $\{f_n(x)g_n(x)\}$ converge uniformly on the interval $[a,b]$? Prove it or give a counterexample!

(b)[10] Assume that $\int_0^\infty \frac{\sin x}{x}dx = \frac{\pi}{2}$. Evaluate the integral $\int_0^\infty \frac{\sin x \cos xy}{x}dx$.

4. **(a)**[10] Prove that the surface area of a sphere with radius r is $S = 4\pi r^2$.

(b)[10] Prove that the volume of a torus with radii a and b are $2\pi^2 a^2 b$.

5. **(a)**[8] Please state the definitions of an open set, a closed set, a connected set, and a compact set in a metric space.

(b)[6] For the following statement, state your reason why you think it is right or wrong.

There is a continuous function $f : \mathbf{R}^2 \mapsto \mathbf{R}^2$ such that $f(A) = B$ where $A = \{(x,y)\,|\,0 < y < 1\}$ and $B = \{(x,y)\,|\,x > 1 \text{ or } x < -1\}$.

(c)[6] For the following statement, state your reason why you think it is right or wrong.

There is a continuous function $f : \mathbf{R}^2 \mapsto \mathbf{R}^2$ such that $f(A) = B$ where $A = \{(x,y)\,|\,x^2+y^2 \leq 1\}$.

7106A