Notice. Read the following definitions before you work on any of the problems. Give details of your work to get credits.

I. Notations and definitions

In the following problem set, the symbols $\mathbb{R}$ and $\mathbb{C}$ are reserved for the fields of all real and complex numbers, respectively.

The symbol K denotes a field, and V and W finite dimensional vector spaces over K. Let $L(V,V)$ be the set of all linear transformations from V to V. For any linear transformation $f : V \to W$, $\ker f$ is the kernel of f and $\operatorname{Im} f$ is the image of f. The set of all polynomials with coefficients in $\mathbb{R}$ having degree no more than 2 is denoted by $P_2(\mathbb{R})$.

The letter n denotes a natural numbers, and $\mathrm{Mat}_n(K)$ is defined to be the set of all n by n matrices over K. We call two matrices $A, B \in \mathrm{Mat}_n(K)$ *similar* if there exists an invertible matrix $P \in \mathrm{Mat}_n(K)$ such that $P^{-1}AP = B$.

We call a linear transformation $f \in L(V,V)$ *cyclic* if there exists some $v \in V$ such that V is spanned by v, $f(v)$, $f^2(v)$, $\ldots$, $f^{n-1}(v)$.

II. Problems

(1) The vectors $v_1 = (1,1,1)$, $v_2 = (1,1,-1)$ and $v_3 = (1,-1,-1)$ form a basis of the vector space $\mathbb{C}^3$. Let $\{u_1, u_2, u_3\}$ be a dual basis of $\{v_1, v_2, v_3\}$ and let $v = (0,1,0) \in \mathbb{C}^3$. Find the inner products $\langle v, u_1\rangle$, $\langle v, u_2\rangle$ and $\langle v, u_3\rangle$. (8%)

(2) Find the conditions so that the following matrix (over $\mathbb{C}$) is diagonalizable. Also find the change-of-coordinate matrix which make it diagonalized. (12%)

$$B = \begin{pmatrix} a & 1 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 1 \\ 0 & 0 & 0 & d \end{pmatrix}.$$

(3) Let $y_0, y_1, y_2, \cdots \in \mathbb{R}$ be the sequence of the Fibonacci numbers where $y_0 = 0$, $y_1 = 1$ and $y_{n+1} = y_n + y_{n-1}$ for all $n \geq 2$. Let $z_n = y_{n-1}$ for $n \geq 1$. Then the Fibonacci sequence can be written as a first order recurrence system (14%)

$$\begin{aligned} y_{n+1} &= y_n + z_n, \\ z_{n+1} &= y_n \end{aligned}$$

with initial conditions $y_1 = 1$ and $z_1 = 0$. By setting $\mathbf{y}_n = (y_n, z_n)^t$ and $A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$, one obtain

$$\mathbf{y}_{n+1} = A\mathbf{y}_n.$$

Now, diagonalize A and obtain a formula for the $(n+1)$-th Fibonacci number y_n.

(4) Let $A, B \in \mathrm{Mat}_n(K)$ with A invertible. Show that the matrix $A + rB$ is invertible for all but finite number of $r \in K$. (14%)

(5) Compute the minimal polynomial for each of the following linear functions, and determine which of them are diagonalizable (Give your reasons!) (12%)

(a) $T : P_2(\mathbb{R}) \to P_2(\mathbb{R})$, where $T(f) = f' + 2f$.

(b) $T : M_n(\mathbb{R}) \to M_n(\mathbb{R})$, where $T(A) = A^t$, the transpose of A.

(6) Let $A = (a_{ij}) \in M_n(F)$ be defined by $a_{ij} = 1 - \delta_{ij}$, where $\delta_{ij} = 1$ if $i = j$, otherwise $\delta_{ij} = 0$. Show that $\det A = (n-1)(-1)^{n-1}$. (10%)

(7) Let $f \in L(V, V)$. Show that if f^2 is cyclic, so is f. Is the converse true? Explain. (15%)

(8) Let $f, g \in L(V, V)$. Suppose that f is cyclic. Show that $f \circ g = g \circ f$ if and only if $g = p(f)$ for some polynomial $p(x) \in K[x]$. (15%)